Winter 2020 Math 126
Topics in Applied Mathematics
Data-driven Uncertainty Quantification

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Lecture 12: Brownian Motion
12.1 Brownian Motion

Definition. Brownian motion (or Wiener process) is a stochastic process \( w(\omega, t), \omega \in \Omega, 0 \leq t \leq 1 \), that satisfies the following four axioms:

1. \( w(\omega, 0) = 0 \) for all \( \omega \).
2. For each \( \omega \), \( w(\omega, t) \) is a continuous function of \( t \).
3. For each \( 0 \leq s \leq t \), \( w(\omega, t) - w(\omega, s) \) is a Gaussian variable with mean zero and variance \( t - s \).
4. \( w(\omega, t) \) has independent increments; i.e., if \( 0 \leq t_1 < t_2 < \cdots < t_n \) then \( w(\omega, t_i) - w(\omega, t_{i-1}) \) for \( i = 1, 2, \ldots, n \) are independent.

12.1 Brownian Motion

Properties of Brownian Motion.

1. The correlation function of Brownian motion is
   \[ E[w(t_1)w(t_2)] = \min(t_1, t_2). \]

2. The Brownian path \( w(\omega, t) \) for a given \( \omega \) is nowhere differentiable with probability 1 with respect to \( t \).
12.1 Brownian Motion

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   \( E[w(t_1)w(t_2)] = \min(t_1, t_2) \).

   **Idea of Proof.** Assume \( t_2 > t_1 \),
   
   \[
   E[w(t_1)w(t_2)] = E[w(t_1)(w(t_2) - t_1) + w(t_1)] \\
   = E[w(t_1)(w(t_2) - w(t_1)) + E[w(t_1)w(t_1)] \\
   = E[w(t_1)w(t_1)] = t_1
   \]

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12.1 Brownian Motion

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\]

2. The Brownian path \( w(\omega, t) \) for a given \( \omega \) is nowhere differentiable with probability 1 with respect to \( t \).

**Idea of Proof.** \( \frac{w(\omega, t + \Delta t) - w(\omega, t)}{\Delta t} \) is Gaussian with mean zero and variance \((\Delta t)^{-1}\).
White Noise. Although the Brownian motion does not have a derivative in the standard sense, it does have a derivative in a distribution sense

\[ v(\omega, t) = \frac{dw(\omega, t)}{dt} = w'(\omega, t) \]

where \( \int_{t_1}^{t_2} v(\omega, s)ds = w(\omega, t_2) - w(\omega, t_1) \). The derivative \( v(\omega, t) \) is called white noise.
12.2 Heat Equation

We want to solve the heat equation

\[ v_t = \frac{1}{2} v_{xx}, \quad v(x, 0) = \phi(x), \quad x \in \mathbb{R}, \quad t > 0, \]

which is a parabolic partial differential equation (PDE).

▶ By Fourier transform,

\[ v(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \hat{v}(k, t) \, dk, \]

\[ v_x(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ike^{ikx} \hat{v}(k, t) \, dk, \]

\[ v_{xx}(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (ik)^2 e^{ikx} \hat{v}(k, t) \, dk, \]

\[ v_t(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \partial_t \hat{v}(k, t) \, dk, \]
12.2 Heat Equation

- Inserting these terms into the heat equation, we have

\[ \partial_t \hat{v}(k, t) = -\frac{1}{2} k^2 \hat{v}(k, t), \]

\[ \hat{v}(k, 0) = \hat{\phi}(k). \]

- The solution to the above ODE is

\[ \hat{v}(k, t) = e^{-\frac{1}{2}k^2t} \hat{\phi}(k) \]

- Thus, the solution \( v(x, t) \) is given by

\[
v(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} e^{-\frac{1}{2}k^2t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx'} \phi(x') dx' dk
\]

\[
= \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-x')^2}{2t}}}{\sqrt{2\pi t}} \phi(x') \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2} \left( k\sqrt{t} - i \frac{x-x'}{\sqrt{t}} \right)^2}}{\sqrt{2\pi}} dk \sqrt{t} dx'
\]
12.2 Heat Equation

(cont’d)

\[= \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-x')^2}{2t}}}{\sqrt{2\pi t}} \phi(x') dx'
\]

\[= \int_{-\infty}^{\infty} \frac{e^{-\frac{x'^2}{2t}}}{\sqrt{2\pi t}} \phi(x + x') dx'
\]

\[= G \ast \phi
\]

where \(G(x) = \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}\) is the Green function of the heat equation.

As the Green function \(G(x)\) is a probability density of a random variable \(w\) with mean zero and variance \(t\),

\[v(x, t) = E[\phi(x + w(\omega, t))].\]
12.3 Heat Equation by Random Walks

We want to approximate the solution of the heat equation on a grid.

▶ Discretize $x$ and $t$ into $x_i = ih$, $t^j = jk$, $i \in \mathbb{Z}$, $j \in \mathbb{N} \cup \{0\}$ with $t = nk$.

▶ We want to find a discrete function $V_i^n$ that approximates $v(ih, nk) = v_i^n$.

▶ In Numerical Analysis, we learn that the solution $V_i^n$ of the following difference equation

$$
\frac{V_i^{n+1} - V_i^n}{k} = \frac{1}{2} \frac{V_{i+1}^n + V_{i-1}^n - 2V_i^n}{h^2}
$$

converges to $v_i^n$ as $h, k \to 0$ and $\lambda := \frac{1}{2} \frac{k}{h^2} \leq \frac{1}{2}$ (that is, the difference scheme is stable and consistent).
12.3 Heat Equation by Random Walks

Choose $\lambda = 1/2$, that is, $k = h^2$. Then the difference equation becomes

$$V_{i}^{n+1} = \frac{1}{2} \left( V_{i+1}^{n} + V_{i-1}^{n} \right).$$

By iterating backward in time and using the notation $V_{i}^{0} = \phi(ih)$, we have

$$V_{i}^{n} = \frac{1}{2} V_{i+1}^{n-1} + \frac{1}{2} V_{i-1}^{n-1}$$

$$= \frac{1}{4} V_{i-2}^{n-2} + \frac{2}{4} V_{i}^{n-2} + \frac{1}{4} V_{i+2}^{n-2}$$

$$= \sum_{j=0}^{n} C_{j,n} \phi((-n + 2j + i)h)$$

where $C_{j,n} = \frac{1}{2^n} \binom{n}{j}$. 
12.3 Heat Equation by Random Walks

Let $\eta_k, k = 1, 2, 3, \ldots, n$ be a random walk

$$\eta_k = \begin{cases} h & \text{probability } 1/2, \\ -h & \text{probability } 1/2 \end{cases}$$

$C_{j,n} = Pr \left( \sum_{k=1}^{n} \eta_k = (-n + 2j)h \right)$.

Using the Central limit theorem, $\sum_{k} \eta_k$ converges to a Gaussian variable with mean 0 and variance $nh^2 = nk = t$ as $n \to \infty$.

Thus,

$$C_{j,n} \sim \frac{e^{-(x')^2/2t}}{\sqrt{2\pi t}} 2h$$

where $x' = (-n + 2j)h$.

Finally,

$$V_i^n \to \int_{-\infty}^{\infty} \frac{e^{-(x-x')^2/2t}}{\sqrt{2\pi t}} \phi(x') dx'$$

as $n \to \infty$. 
12.4 Wiener Measure

We showed that the solution of the heat equation \( v(x, t) \) can be written as

\[
v(x, t) = E[\phi(x + w(\omega, t))]\]

where \( \phi(x) \) is the initial value.

The expectation is on the sample space, the space of Brownian motions. What is the probability with respect to \( w(\omega, t) \)? How do we define a probability distribution on \( w(\omega, t) \)?

The difficulty is that the Brownian motion is in an infinite-dimensional space.
12.4 Wiener Measure

- We consider the space of continuous functions \( u(t) \) such that \( u(0) = 0 \). This is our sample space \( \Omega \).
- Pick an instant in time, say \( t_1 \), and associate with this instant a window of values \((a_1, b_1]\), where \(-\infty \leq a_1, b_2 \leq \infty\).
- Consider a subset of all continuous functions that pass through this window and denote it by \( C_1 \) (called a cylinder set).
- For every instant and every window, we can define a corresponding cylinder set, i.e., \( C_i \) is the subset of all continuous functions that pass through the windows \((a_i, b_i]\) at the instant \( t_i \).
- Consider two cylinder sets, \( C_1 \) and \( C_2 \). Then \( C_1 \cap C_2 \) is the set of functions that pass through both windows. Similarly, \( C_1 \cup C_2 \) is the set of functions that pass through either \( C_1 \) or \( C_2 \).
- This forms an algebra (closed under finite disjoint unions, intersections, and complements).
12.4 Wiener Measure

▶ The probability measure of $C_1$ is defined as

$$Pr(C_1) = \int_{a_1}^{b_1} \frac{e^{-s_1^2/2t_1}}{\sqrt{2\pi t_1}} ds_1.$$ 

▶ There exists a $\sigma$-algebra and a probability measure $dW$ (Wiener measure) that extends the probability on the cylinder sets.

Example.

$$v(x, t) = E[\phi(x + w(\omega, t))] = \int \phi(x + w(\omega, t))dW$$

$$= \int_{-\infty}^{\infty} \phi(x + x') \frac{e^{-(x')^2/2t}}{\sqrt{2\pi t}} dx'$$
12.4 Wiener Measure

**Example.** \( \int w^2(\omega, 1) dW = \int_{-\infty}^{\infty} u^2 e^{-u^2/2} \frac{1}{\sqrt{2\pi}} = 1 \)

**Example.** Assume that we can extend Fubini’s theorem, that is, we can change the order of integrations. We want to find the expected value of \( \int_0^1 w^2(\omega, s) ds \) for the Brownian motion \( w \).

\[
E[\int_0^1 w^2(\omega, s) ds] = \int_0^1 w^2(\omega, s) ds dW
\]

\[
= \int_0^1 ds \int dW w^2(\omega, s) = \int_0^1 s ds = \frac{1}{2}.
\]
12.4 Wiener Measure

**Example.** Find the expected value of $w^2(\omega, 1/2)w^2(\omega, 1)$.

\[
\int w^2(\omega, 1/2)w^2(\omega, 1)\,dW = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2(x+y)^2 \frac{e^{-x^2-y^2}}{\pi} \,dx\,dy = 1.
\]
12.5 Heat Equation with Potential

Now we consider the following heat equation with a potential $U(x)$

$$v_t = \frac{1}{2} v_{xx} + U(x)v, \quad v(x,0) = \phi(x).$$

For $\lambda = \frac{1}{2} \frac{k}{h^2} \leq \frac{1}{2}$, the solution to the following difference equation converges to the solution of the original equation (a good exercise for numerical analysis)

$$\frac{V_{i+1}^n - V_i^n}{k} = \frac{1}{2} \frac{V_{i-1}^n + V_{i+1}^n - 2V_i^n}{h^2} + \frac{1}{2} \left( U_{i-1}V_{i-1}^n + U_{i+1}V_{i+1}^n \right),$$

where $U_i = U(ih)$. 
12.5 Heat Equation with Potential

For $\lambda = 1/2$,

$$V_{i}^{n+1} = \frac{1}{2}(V_{i-1}^{n} - V_{i+1}^{n}) + \frac{k}{2}(U_{i+1}V_{i+1}^{n} + U_{i-1}V_{i-1}^{n})$$

$$= \frac{1}{2}(1 + kU_{i+1})V_{i+1}^{n} + \frac{1}{2}(1 + kU_{i-1})V_{i-1}^{n}.$$ 

By induction, we have

$$V_{i}^{n} = \sum_{l_{1}=\pm 1,\ldots,l_{n}=\pm 1} \frac{1}{2^{n}}(1+kU_{i+l_{1}})\cdots(1+kU_{i+l_{1}+\ldots+l_{n}})V_{i+l_{1}+\ldots+l_{n}}^{0}$$

Let $\eta_{k}, k = 1, 2, 3, \ldots, n$ be a random walk

$$\eta_{k} = \begin{cases} h & \text{probability } 1/2, \\ -h & \text{probability } 1/2 \end{cases}$$

$$Pr(\eta_{1} = l_{1}h, \ldots, \eta_{n} = l_{n}h) = \frac{1}{2^{n}}$$
12.5 Heat Equation with Potential

- A probabilistic interpretation of the solution $V^i_n$ is

$$V^i_n = E_{\forall \text{paths}} \{ \prod_{m=1}^{n} (1 + kU(ih + \eta_1 + \cdots + \eta_m)) \} \times \phi(ih + \eta_1 + \cdots + \eta_n) \}$$

- Let $\tilde{w}(s)$ be the path connecting $\eta_i$ linearly for $0 \leq s \leq t$. Then we have

$$V^i_n = E_{\forall \text{broken line paths}} \{ \prod_{m=1}^{n} (1 + kU(ih + \tilde{w}(s_m))) \} \times \phi(ih + \tilde{w}(t)) \}$$

where $s_m = mk$.

- For $k|U| < 1/2$, $(1 + kU) = \exp(kU + \epsilon)$ where $|\epsilon| \leq Ck^2$.

- $\prod_{m=1}^{n} (1 + kU(ih + \tilde{w}(s_m))) = \exp(k \sum_{m=1}^{n} U(ih + \tilde{w}(s_m)) + \epsilon')$, where $|\epsilon'| \leq nCk^2 = Ctk$. 

12.5 Heat Equation with Potential

- \( V_i^n = E_{\text{all broken line paths}} \left\{ e^{\int_0^t U(ih+\tilde{w}(s))ds} \phi(ih + \tilde{w}(t)) \right\} + \text{small terms.} \)

- As \( h \) and \( k \) tend to zero, the broken line paths \( ih + \tilde{w}(s) \) look more and more like Brownian motion paths \( ih + w(s) \), so in the limit,

\[
\nu(x, t) = E_{\text{all Brownian motion paths}} \left\{ e^{\int_0^t U(x+w(s))ds} \phi(x + w(t)) \right\}
\]

\[
= \int dW e^{\int_0^t U(x+w(s))ds} \phi(x + w(t)),
\]

the **Feynman-Kac** formula!
12.5 Heat Equation with Potential

**Feynman diagrams.** We introduce an $\epsilon$, a small parameter, in front of the potential $U$. After expanding in a Taylor series of $\epsilon$,

$$e^\int_0^t \epsilon U(x+w(s))ds = 1 + \epsilon \int_0^t U(x+w(s))ds + \frac{1}{2} \epsilon^2 \left( \int_0^t U(x + w(s))ds \right)^2 + \cdots$$

- constant term $T_0 = \int dW \phi(x + w(t))$
  
  $$= \int_{-\infty}^{\infty} e^{-\frac{(x-z)^2}{2t}} \frac{1}{\sqrt{2\pi t}} \phi(z)dz = \int_{-\infty}^{\infty} K(x - z, t)\phi(z)dz$$
  
  with the vacuum propagator $K(z, s) = \frac{1}{\sqrt{2\pi s}} e^{-z^2/2s}$

- $\epsilon$-order term $T_1$

  $$T_1 = \epsilon \int dW \int_0^t U(x + w(s))\phi(x + w(t))ds$$
  
  $$= \epsilon \int_0^t ds \int dW U(x + w(s))\phi(x + w(t)).$$
12.5 Heat Equation with Potential

▶ $T_1$ cont’d

$$T_1 = \epsilon \int_0^t ds \int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{\infty} dz_2 K(z_1 - x, s) \cdot U(z_1) K(z_2, t - s) \phi(z_1 + z_2).$$

▶ Similarly (straightforward but not easy), the $\epsilon^2$ term

$$T_2 = \epsilon^2 \int_0^t dt_2 \int_0^{t_2} dt_1 \int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{\infty} dz_2 \int_{-\infty}^{\infty} \cdot K(z_1 - x, t_1) U(z_1) K(z_2, t_2 - t_1) U(z_1 + z_2) \cdot K(z_3, t - t_2) \phi(z_1 + z_2 + z_3).$$