Winter 2020 Math 126
Topics in Applied Mathematics
Data-driven Uncertainty Quantification

Yoonsang Lee (yoonsang.lee@dartmouth.edu)

Lecture 13: Stochastic Differential Equations

13.1 Stochastic Differential Equations

$$\frac{du}{dt} = a(t, u) + \frac{dw}{dt}, \quad a \text{ is smooth} \tag{1}$$

where $w(\omega, t)$ is Brownian motion.

- One type of random systems; other random forcing terms are available.
- Given $u(0)$, an initial value, what does this equation mean?
- $\frac{dw}{dt}$ is defined in the distribution sense, i.e., $\int_{t_1}^{t_2} dw = w(t_2) - w(t_1)$.
- For any $t_2 > t_1 \geq 0$

$$u(t_2) - u(t_1) = \int_{t_1}^{t_2} a(t, u(t))dt + w(t_2) - w(t_1).$$

- Note that $w(t_2) - w(t_1)$ is a Gaussian variable with mean zero and variance $t_2 - t_1$.

13.1 Stochastic Differential Equations

The following formulation is equivalent to Eq (1)

$$du = a(t, u)dt + dw \tag{2}$$

Example. u : velocity, a : deterministic force, $\frac{dw}{dt}$: random (white noise) force.

- If $a = 0$, $du = dw$, that is, u is Brownian motion.
- For a small $k > 0$,

$$\boxed{u((n+1)k) = u(nk) + a(nk, u(nk))k + w(k)} \tag{3}$$

where $w(k) \sim N(0, k)$. This is the Euler-Maruyama method.

13.2 Stochastic Integration

More general class of SDE is

$$du = a(t, u(t))dt + b(t, u(t))dw \tag{4}$$

The meaning of the above SDE is

$$u(t_2) - u(t_1) = \int_{t_1}^{t_2} a(t, u(t))dt + \int_{t_1}^{t_2} b(t, u(t))dw$$

for any $t_2 > t_1 \geq 0$. The first integral is well-defined. However, the second integral is not clear.

Note. $\int u(t)dW$ and $\int u(t)dw$ are different. The first one is an integration with respect to the Wiener measure, that is, it considers all possible Brownian paths, while the second one is an integration with respect to one Brownian motion.

13.2 Stochastic Integration

- The stochastic integral is defined as

$$\int_0^t b(s, u(s))dw = \lim_{\sup|t_{i+1}-t_i|\to 0} \sum_{i=0}^{m-1} b_i\{w(t_{i+1} - w(t_i)\}$$

  where $\{t_i\}$ is a partition of $[0, t]$, i.e.,
  $0 = t_0 < t_1 < \cdots < t_m = t$.
- There are several methods for stochastic integration depending on how to choose b_i
  1. **Ito** integral: $b_i = b(t_i, u(t_i))$
  2. **Stratonovich** integral: $b_i = \frac{1}{2}(b(t_i, u(t_i) + b(t_{i+1}, u(t_{i+1}))$

13.2 Stochastic Integration

Example. Calculate $\int_0^t w dw$.

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Heuristic explanation. First guess (assuming that w is a deterministic bounded variation function) is $\frac{1}{2}w^2(t)$.

- Using Ito integral,

$$\int_0^t w dw \approx \sum_{i=0}^{m-1} w(t_i)\left(w(t_{i+1}) - w(t_i)\right)$$

Thus, $E[\int_0^t w dw] = 0$ and it is natural to guess

$$\boxed{\text{Ito integral: } \int_0^t w dw = \frac{1}{2}w^2(t) - \frac{1}{2}t.}$$

13.2 Stochastic Integration

Example. Calculate $\int_0^t w dw$.

Heuristic explanation. First guess (assuming that w is a deterministic bounded variation function) is $\frac{1}{2}w^2(t)$.

- Using Stratonovich integral,

$$\int_0^t w dw \approx \sum_{i=0}^{m-1} \frac{1}{2}\left(w(t_{i+1} + w(t_i)\right)\left(w(t_{i+1}) - w(t_i)\right)$$

$$= \sum_{i=0}^{m-1} \frac{1}{2}\left(w^2(t_{i+1}) - w^2(t_i)\right) = \frac{1}{2}w^2(t)$$

Thus, $E[\int_0^t w dw] = \frac{1}{2}t$ and it is natural to guess

$$\boxed{\text{Stratonovich integral: } \int_0^t w dw = \frac{1}{2}w^2(t).}$$

13.3 Fokker-Planck Equations

Definition. A stochastic process $u(\omega, t), t \in \mathbb{R}$ or $\mathbb{R}^+$ is called **Markov process** if

$$E[u(\omega, t')|u(\omega, s), s \leq t] = E[u(\omega, t')|u(\omega, t)]$$

- By construction, the solution to the SDE is a Markov process.
- Let $p(x, t)$ be the probability density of $u(\omega, t)$ at time t, that is,
$$Pr(x < u(t) \leq x + dx) = p(x, t)dx$$
- Chapman-Kolmogorov equation
$$p(x, t + k) = \int p(x + y, t)\psi(x, y, k)dy$$
where ψ is the transition probability that the value of u changes from $x + y$ at time t to x at time $t + k$.

13.3 Fokker-Planck Equations

For our discussion, we will consider the **Langevin** equation (or **Ornstein-Uhlenbeck** equation)

$$du = -audt + dw, \quad a > 0 \tag{5}$$

We want to derive an equation satisfied by $p(x, t)$ called **Fokker-Planck** equation (or **Kolmogorov** equation).

- For a small $k > 0$ and $u^n = u(nk)$ with $ak < 1$,

$$u^{n+1} - u^n = -aku^n + w^{n+1} - w^n$$

- $u^{n+1} - u^n + aku^n$ is Gaussian with mean zero and variance k

$$Pr(x < u^{n+1} \leq x + dx) = \frac{1}{\sqrt{2\pi k}} \exp\left(-\frac{(x - u^n + aku^n)^2}{2k}\right)$$

13.3 Fokker-Planck Equations

- Using a notation $u^n = x + y$,

$$p(x, t+k) = \int_{-\infty}^{\infty} p(x+y, t) \frac{1}{\sqrt{2\pi k}} \exp\left(-\frac{(-y + ak(x+y))^2}{2k}\right) dy$$

- After rearranging the exponent,

$$p(x, t+k) = \int_{-\infty}^{\infty} p(x+y, t) \frac{1}{\sqrt{2\pi k}} \exp\left(-\frac{((1-ak)y - akx)^2}{2k}\right) dy$$

- Expand $p(x+y, t)$ around x,

$$p(x+y, t) = p(x, t) + y p_x(x, t) + \frac{y^2}{2} p_{xx}(x, t) + \frac{y^3}{6} p_{xxx}(x, t) + \mathcal{O}(y^4)$$

13.3 Fokker-Planck Equations

- $I_1 = \int_{-\infty}^{\infty} p(x,t) \frac{1}{\sqrt{2\pi k}} \exp\left(-\frac{((1-ak)y-akx)^2}{2k}\right) dy$
- $I_2 = \int_{-\infty}^{\infty} y p_x(x,t) \frac{1}{\sqrt{2\pi k}} \exp\left(-\frac{((1-ak)y-akx)^2}{2k}\right) dy$
- $I_3 = \int_{-\infty}^{\infty} \frac{y^2}{2} p_{xx}(x,t) \frac{1}{\sqrt{2\pi k}} \exp\left(-\frac{((1-ak)y-akx)^2}{2k}\right) dy$
- $I_4 = \int_{-\infty}^{\infty} \frac{y^3}{6} p_{xxx}(x,t) \frac{1}{\sqrt{2\pi k}} \exp\left(-\frac{((1-ak)y-akx)^2}{2k}\right) dy$
- For $z = (1-ak)y$, I_1 becomes

$$I_1 = p(x,t) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi k}} \exp\left(-\frac{(z-akx)^2}{2k}\right) \frac{dz}{1-ak}$$

$$= \frac{p(x,t)}{1-ak} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi k}} \exp\left(-\frac{(z-akx)^2}{2k}\right) dz$$

$$= \frac{p(x,t)}{1-ak} = p(x,t)(1+ak+\mathcal{O}(k^2)) = \boxed{p(x,t)(1+ak)+\mathcal{O}(k^2)}$$

13.3 Fokker-Planck Equations

- For the same change of variable,

$$I_2 = p_x(x,t)\int_{-\infty}^{\infty}\frac{z}{1-ak}\frac{1}{\sqrt{2\pi k}}\exp\left(-\frac{(z-akx)^2}{2k}\right)\frac{dz}{1-ak}$$

$$= \frac{p_x(x,t)}{(1-ak)^2}akx = p_x(x,t)(1+2ak+\mathcal{O}(k^2))akx$$

$$= \boxed{p_x(x,t)akx + \mathcal{O}(k^2)}$$

- Similarly,

$$I_3 = p_{xx}(x,t)\int_{-\infty}^{\infty}\frac{z^2}{2(1-ak)^2}\frac{\exp\left(-\frac{(z-akx)^2}{2k}\right)}{\sqrt{2\pi k}}\frac{dz}{1-ak}$$

$$= p_{xx}(x,t)\frac{k+(akx)^2}{2(1-ak)^3} = \boxed{p_{xx}(x,t)\frac{k}{2} + \mathcal{O}(k^2)}$$

13.3 Fokker-Planck Equations

- Also,

$$I_4 = p_{xxx}(x,t)\int_{-\infty}^{\infty}\frac{z^3}{6(1-ak)^3}\frac{\exp\left(-\frac{(z-akx)^2}{2k}\right)}{\sqrt{2\pi k}}\frac{dz}{1-ak}$$

$$= p_{xxx}(x,t)\frac{(3axk^2+(akx)^3)}{6(1-ak)^3}$$

$$= \boxed{p_{xxx}(x,t)\mathcal{O}(k^2)}$$

- Finally, using I_1, I_2, I_3 and I_4, we have

$$p(x,t+k) = p(x,t) = p(x,t)ak + p_x(x,t)akx + \frac{k}{2}p_{xx}(x,t) + \mathcal{O}(k^2)$$

$$\Rightarrow \frac{p(x,t+k)-p(x,t)}{k} = p(x,t)a + p_x(x,t)ax + \frac{1}{2}p_{xx}(x,t) + \mathcal{O}(k)$$

13.3 Fokker-Planck Equations

After taking $k \to 0$, we have that the Fokker-Planck equation of the Langevin equation

$$du = -audt + dw$$

is

$$\boxed{p_t(x,t) = \partial_x(axp(x,t)) + \frac{1}{2}p_{xx}(x,t)}$$

13.3 Fokker-Planck Equations

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Exercise. Derive the Fokker-Planck equation of the following differential equations

1. $du = -audt + \sigma dw$
2. $du = -audt$

13.3 Fokker-Planck Equations

Example. Random walk with killing

- Let $U(x,t)$ be a smooth function of x and t and $0 \le U \le A$.
- Choose $k > 0$ so that $1 - kA \ge 0$.
- Let η be the standard random walk, i.e.,
  $$\eta = \begin{cases} h & \text{w/ } 1/2 \\ -h & \text{otherwise} \end{cases}$$
- Let u^n be a random walk such that $u(0) = 0$ and
  $$u^{n+1} = \begin{cases} u^n + \eta & \text{with probability } 1 - kU(u^n, t) \\ \text{killed} & \text{otherwise} \end{cases}$$

  Then
  $$p(x, t+k) = \int_{-\infty}^{\infty} p(x+y, t)(1 - U(x+y, t)k) \frac{\exp\left(-y^2/2k\right)}{\sqrt{2\pi k}} dy$$

13.3 Fokker-Planck Equations

Example. Random walk with killing

- Let $I_1 = \int_{-\infty}^{\infty} p(x+y,t)\frac{\exp\left(-y^2/2k\right)}{\sqrt{2\pi k}}dy$, then using similar calculations as before, we have

$$I_1 = p(x,t) + \frac{k}{2}p_{xx} + \mathcal{O}(k^2)$$

- For $I_2 = \int_{-\infty}^{\infty} p(x+y,t)U(x+y,t)\frac{\exp\left(-y^2/2k\right)}{\sqrt{2\pi k}}dy$, we have

$$I_2 = U(x,t)p(x,t) + \mathcal{O}(k^2).$$

- Thus,

$$p(x,t+k) = p(x,t) + \frac{k}{2}p_{xx}(x,t) - kU(x,t)p(x,t) + \mathcal{O}(k^2)$$

$$\Rightarrow \boxed{p_t = \frac{1}{2}p_{xx} - U(x,t)p(x,t)}$$

Homework

1. Let $u \in \mathbb{R}$ be Brownian motion

$$du = dw.$$

   1.1 Using the Monte Carlo simulation, estimate the density of u at $t = 1$. Use the Euler-Maruyama scheme to solve the SDE.
   1.2 For the same SDE, restrict u to be in $[-1, 1]$. That is, if u is not in $[-1, 1]$, it is removed. Calculate the probability to reach $t = 1$ without being removed.

2. Let $u \in \mathbb{R}$ be a stochastic process

$$du = -\frac{1}{2}u(1 - u^2)dt + \frac{1}{10}dw.$$

   2.1 Estimate the density of u at $t = 1$ when the initial density is $\delta(x)$.
   2.2 Estimate the density of u at $t = 1$ when the initial density is Gaussian with mean -0.1 and variance 0.1^2.

3. Derive the Fokker-Planck equation of $du = -adt + dw, u \in \mathbb{R}$ for a constant a.