Winter 2020 Math 106
Topics in Applied Mathematics
Data-driven Uncertainty Quantification

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Data Assimilation
Lecture 17: Kalman Filter

17.1 Data Assimilation

k: index for time $t = k\Delta t$ for a time interval Δt.

- a system of interest with uncertainty

$$u_k = f(u_{k-1}) + \sigma_d \xi_{k-1}$$

$$\xi_k \sim N(0, \sigma_d^2)$$

- observations available uniformly in time

$$v_k = g(u_k) + \epsilon_k$$

$\epsilon_k \sim N(0, \sigma_0^2)$ observation error

Notation. $u_{1:k} = \{u_1, u_2, ..., u_k\}$, $v_{1:k} = \{v_1, v_2, ..., v_k\}$.
Goal of data assimilation. At $t = k\Delta t$, we want to estimate u_k using $v_{1:k}$.

$$\boxed{p(u_k|v_{1:k}) = \frac{p(v_k|u_k)p(u_k|v_{1:k-1})}{p(v_k|v_{1:k-1})}}$$

17.1 Data Assimilation

$$p(u_k|v_{1:k}) = \frac{p(v_k|u_k)p(u_k|v_{1:k-1})}{p(v_k|v_{1:k-1})}$$

Derivation.

$$\begin{aligned}
p(u_k|v_{1:k}) &= \frac{p(v_{1:k}|u_k)p(u_k)}{p(v_{1:k})} \\
&= \frac{p(v_k, v_{1:k-1}|u_k)p(u_k)}{p(v_{1:k})} \\
&= \frac{p(v_k|v_{1:k-1}, u_k)p(v_{1:k-1}|u_k)p(u_k)}{p(v_{1:k})} \\
&= \frac{p(v_k|v_{1:k-1}, u_k)p(u_k|v_{1:k-1})p(v_{1:k-1})p(u_k)}{p(v_{1:k})p(u_k)} \\
&= \frac{p(v_k|v_{1:k-1}, u_k)p(u_k|v_{1:k-1})p(v_{1:k-1})}{p(v_k|v_{1:k-1})p(v_{1:k-1})} \\
&= \frac{p(v_k|v_{1:k-1}, u_k)p(u_k|v_{1:k-1})}{p(v_k|v_{1:k-1})} = \frac{p(v_k|u_k)p(u_k|v_{1:k-1})}{p(v_k|v_{1:k-1})}
\end{aligned}$$

17.1 Data Assimilation

- $p(u_k|v_{1:k-1})$: prior density of u_k. This is calculated from the previous step posterior density $p(u_{k-1}|v_{1:k-1})$ using one of the methods to propagate uncertainty (MC, gPC, perturbation, etc).
- $p(v_k|u_k)$: likelihood of v_k. Under the Gaussian assumption of the observation error, we have

$$p(v_k|u_k) = \frac{1}{\sqrt{2\pi\sigma_o^2}} \exp\left(-\frac{(v_k - g(u_k))^2}{2\sigma_o^2}\right)$$

- $p(v_k|v_{1:k-1})$: normalization constant.

17.2 Kalman Filter

Example. Scalar linear system $u \in \mathbb{R}$.

$$u_k = au_{k-1} + \xi_{k-1}, \quad \xi_{k-1} \sim N(0, \sigma_d^2)$$

$$v_k = u_k + \epsilon_k, \quad \epsilon_k \sim N(0, \sigma_o^2)$$

- Assume that $u_{k-1}|v_{1:k-1}$ is Gaussian with mean m_{k-1} and variance C_{k-1}^2, which are the mean and variance of the previous step posterior density $p(u_{k-1}|v_{1:k-1})$.
- Then $p(u_k|v_{1:k-1})$ is also Gaussian with mean $\tilde{m}_k$ and variance $\tilde{C}_k^2$

$$\tilde{m}_k = am_{k-1}$$

$$\tilde{C}_k^2 = a^2 C_{k-1}^2 + \sigma_d^2$$

17.2 Kalman Filter

Example. Scalar linear system $u \in \mathbb{R}$.

$$u_k = a u_{k-1} + \xi_{k-1}, \quad \xi_{k-1} \sim N(0, \sigma_d^2)$$

$$v_k = u_k + \epsilon_k, \quad \epsilon_k \sim N(0, \sigma_o^2)$$

- The posterior $p(u_k | v_{1:k})$ is also Gaussian with mean m_k and variance C_k^2

$$m_k = \frac{\tilde{m}_k \sigma_o^2 + v_k \tilde{C}_k^2}{\tilde{C}_k^2 + \sigma_o^2}$$

$$C_k^2 = \frac{\tilde{C}_k^2 \sigma_o^2}{\tilde{C}_k^2 + \sigma_o^2}$$

Idea of Proof. Match

$$-\frac{(u_k - \tilde{m}_{k-1})^2}{2\tilde{C}_k^2} - \frac{(v_k - u_k)^2}{2\sigma_o^2} = -\frac{(u_k - m_k)^2}{2C_k^2}$$

17.2 Kalman Filter

Example. Scalar linear system $u \in \mathbb{R}$.

$$u_k = a u_{k-1} + \xi_{k-1}, \quad \xi_{k-1} \sim N(0, \sigma_d^2)$$

$$v_k = u_k + \epsilon_k, \quad \epsilon_k \sim N(0, \sigma_o^2)$$

- For consistency with another formula we will discuss later, the mean and variance the following representation

$$\boxed{m_k = \tilde{m}_k + K(v_k - \tilde{m}_k)}$$

$$\boxed{C_k^2 = (1 - K)\tilde{C}_k^2}$$

where $K = \frac{\tilde{C}_k^2}{\tilde{C}_k^2 + \sigma_o^2}$ is called "Kalman gain".

17.2 Kalman Filter

Kalman filter for a d-dimensional linear system. For $u \in \mathbb{R}^d$

$$u_k = Au_{k-1} + \xi_{k-1}, \quad \xi \sim N(0, \Sigma)$$

$$v_k = Hu_k + \epsilon_k, \quad \epsilon \sim N(0, \Gamma)$$

where Σ and Γ are symmetric positive definite matrices.

- The prior mean $\tilde{m}_k$ and covariance $\tilde{C}_k$ are given by

$$\tilde{m}_k = Am_{k-1}$$

$$\tilde{C}_k^2 = AC_{k-1}^2 A^T + \Sigma$$

where m_{k-1} and C_{k-1} are the mean and covariance of the previous step posterior distribution $p(u_{k-1} | v_{1:k-1})$.

17.2 Kalman Filter

Kalman filter for a d-dimensional linear system. For $u \in \mathbb{R}^d$

$$u_k = Au_{k-1} + \xi_{k-1}, \quad \xi \sim N(0, \Sigma)$$

$$v_k = Hu_k + \epsilon_k, \quad \epsilon \sim N(0, \Gamma)$$

where Σ and Γ are symmetric positive definite matrices.

- The posterior mean m_k and covariance C_k^2 are given by

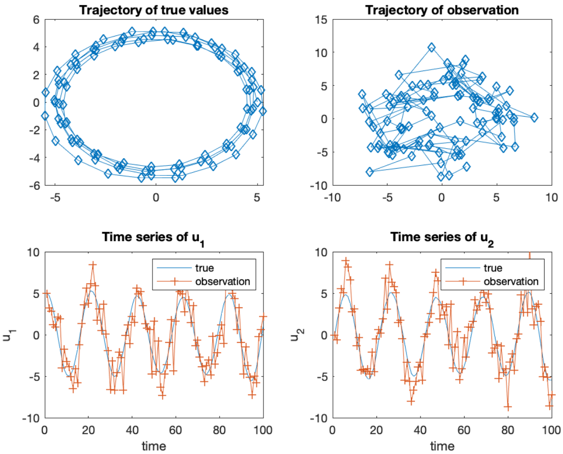
$$\boxed{\begin{aligned} m_k &= \tilde{m}_k + K_k(v_k - H\tilde{m}_k) \\ C_k^2 &= (1 - K_k H)\tilde{C}_k^2 \\ K_k &= \tilde{C}_k^2 H^T \left(H\tilde{C}_k^2 H^T + \Gamma \right)^{-1} \end{aligned}} \quad (1)$$

  where K is the **Kalman gain matrix**.

17.2 Kalman Filter

Example. $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ with $\theta = 0.3$. $\Sigma = \sigma^2 I_2$. $\Gamma = \sigma_o^2 I_2$.

True and noisy observation values.

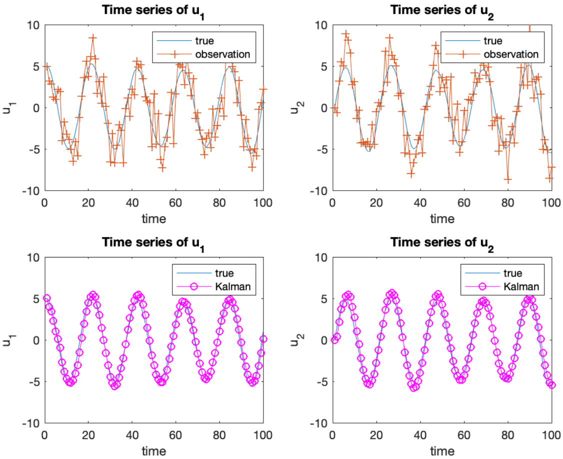


17.2 Kalman Filter

Example. $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ with $\theta = 0.3$. $\Sigma = \sigma^2 I_2$.

$\Gamma = \sigma_o^2 I_2$.

Kalman filtering result.



17.3 Continuous Time Model

Example. One of your homework problem (with different notations)

$$du - \gamma u dt + \sigma dw \tag{2}$$

- It is straightforward to derive an equation for the mean

$$\frac{dE[u]}{dt} = -\gamma E[u] \Rightarrow E[u] = u_0 e^{-\gamma t}$$

- But not straightforward for the variance. Let $u = E[u] + \tilde{u}$. Then

$$\frac{d\tilde{u}}{dt} = -\gamma \tilde{u} + \sigma dw$$

$$\frac{1}{2}\frac{dVar(u)}{dt} = \frac{1}{2}\frac{dE[\tilde{u}^2]}{dt} = E[\tilde{u}\frac{d\tilde{u}}{dt}]$$

$$= E\left[-\gamma u^2 dt + \sigma u dw\right] ??$$

17.3 Continuous Time Model

Example. One of your homework problem (with different notations)

$$du - \gamma u dt + \sigma dw \qquad (2)$$

Another approach using integrating factors and white noise.

- The solution to (2) is given by

$$u(t) = u_0 e^{-\gamma t} + \sigma \int_0^t e^{-\gamma(t-s)} dw(s).$$

- Note that $E[u] = u_0 e^{-\gamma t}$ and $\tilde{u} = \sigma \int_0^t e^{-\gamma(t-s)} dw(s)$.
- $Var(u(t)) = E[\tilde{u}(t)^2]$

$$= E\left[\sigma^2 e^{-2\gamma t} \int_0^t \int_0^t e^{\gamma(t'+s')} v(t') v(s') dt' ds'\right]$$

where $v(t')$ is the white noise of $w(t')$.

17.3 Continuous Time Model

Example. One of your homework problem (with different notations)

$$du - \gamma u dt + \sigma dw \tag{2}$$

Another approach using integrating factors and white noise.

$$= \sigma^2 e^{-2\gamma t} \int_0^t \int_0^t e^{\gamma(t'+s')} E\left[v(t')v(s')\right] dt' ds'$$

$$= \sigma^2 e^{-2\gamma t} \int_0^t \int_0^t e^{\gamma(t'+s')} \delta(t'-s') dt' ds'$$

$$= \sigma^2 e^{-2\gamma t} \int_0^t e^{2\gamma t'} dt'$$

$$= \frac{\sigma^2}{2\gamma}\left(1 - e^{-2\gamma t}\right)$$

17.3 Continuous Time Model

Example. One of your homework problem (with different notations)

$$du - \gamma u dt + \sigma dw \tag{2}$$

Therefore, we have

$$m(t) = E[u(t)] = u_0 e^{-\gamma t}$$

$$C^2(t) = \frac{\sigma^2}{2\gamma}\left(1 - e^{-2\gamma t}\right)$$

Exercise. We assumed that u_0 is a fixed value (not random). What are the mean and variance of $u(t)$ for $u_0 \sim N(m_0, \sigma_0^2)$ where m_0 and σ_0^2 are fixed constants?