NAME : Key

Math 13

Midterm 1
October 4, 2016

Prof. Pantone

INSTRUCTIONS: This is a closed book exam and no notes are allowed. You are not to provide or receive help from any outside source during the exam except that you may ask the instructor for clarification of a problem. You have 120 minutes and you should attempt all problems.

- Print your name in the space provided.
- Calculators or other computing devices are not allowed.
- Except when indicated, you must show all work and give justification for your answer. **A correct answer with incorrect work will be considered wrong.**

All work on this exam should be completed in accordance with the Dartmouth Academic Honor Principle.

Problem	Points	Score
1	6	
2	12	
3	8	
4	8	
5	8	
6	8	
Total	50	

Section 1: True/False.

1. (6) Choose the correct answer. *No justification is required for your answers. No partial credit will be awarded.*

(a) The plane $z = 3$ is expressed in spherical coordinates as $\rho = 3\sec(\varphi)$.

True False

(b) The level curves of $z = 3 - \sqrt{x^2 + y^2}$ are circles.

True False

(c) When converting from Cartesian coordinates to spherical coordinates, we use the substitution $x = \rho\cos(\theta)$, $y = \rho\sin(\theta)$, and $z = \rho\cos(\varphi)$.

True False

(d) The integral of $f(x, y) = 3$ over the region D that is the circle of radius 2 in the xy-plane centered at $(1, 1)$ is 12π.

True False

(e) The polar curves $r = 3 - \theta$ and $r = \cos(\theta)$ intersect at the origin when $\theta = \pi$.

True False

(f) Let D be the region described by $\theta_1 \leq \theta \leq \theta_2$ and $r_1(\theta) \leq r \leq r_2(\theta)$. Suppose $f(x, y)$ is continuous. Then,

$$\iint_D f(x, y)dA = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos(\theta), r\sin(\theta))\, dr\, d\theta.$$

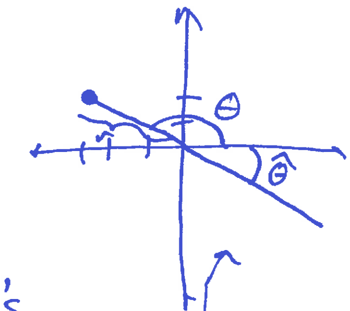
True False

Section 2: Short Answer.

2. (12) *Circle your answer. No justification is required. No partial credit will be awarded.*

(a) What are the polar coordinates of the point $(x, y) = (-3, 2)$?

$r = \sqrt{(-3)^2 + (2)^2} = \sqrt{13}$



$\tan(\theta) = -\frac{2}{3}$

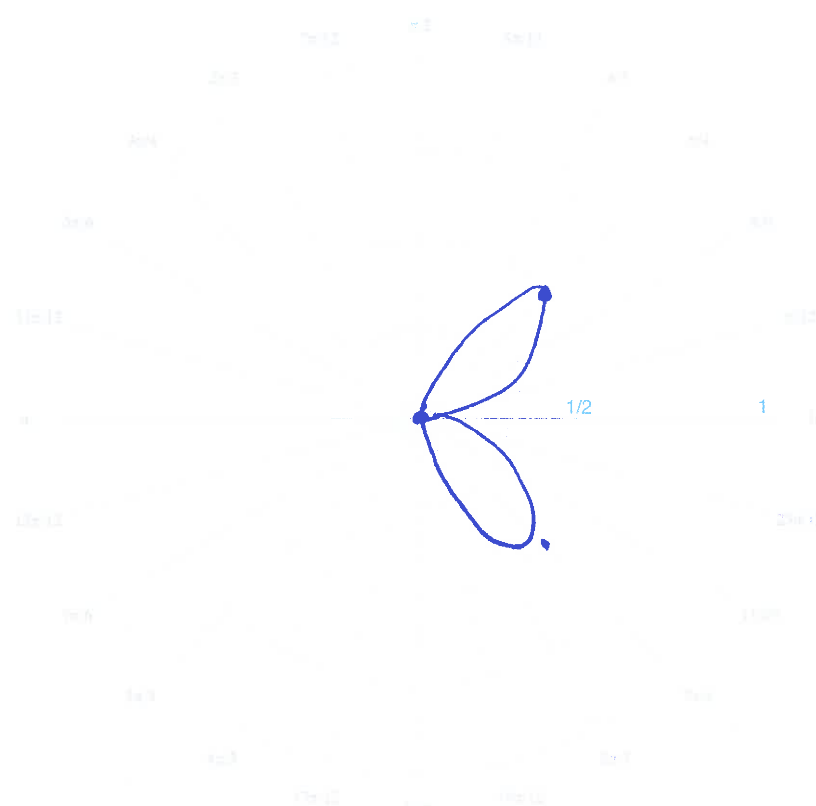
But $\theta \neq \arctan(-\frac{2}{3})$ because that's in Q4, not Q2, so $\arctan(-2/3)$ is $\hat{\theta}$.

To get θ, we want $\pi + \hat{\theta}$.

So, $(r, \theta) = \boxed{(\sqrt{13}, \pi + \arctan(-\frac{2}{3}))}$ or $\boxed{(\sqrt{13}, \pi - \arctan(\frac{2}{3}))}$

(b) Plot the polar graph $r = \cos(\theta)\sin(\theta)$ on the grid below for $0 \le \theta \le \pi$. (*Note this domain restriction! Graphs with θ outside of the range $[0, \pi]$ will be marked incorrect.*)

θ	$\cos\theta \sin\theta$
0	0
$\pi/4$	$1/2$
$\pi/2$	0
$3\pi/4$	$-1/2$
π	0



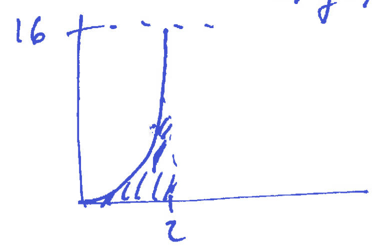
(c) Calculate $\int_0^{16} \int_{y^{1/4}}^{2} e^{x^5}\, dx\, dy$. $\longrightarrow$ y in $[0, 16]$

x in $[y^{1/4}, 2]$

$x = y^{1/4}$

$\Rightarrow y = x^4$

Change the order.



x in $[0, 2]$

y in $[0, x^4]$.

$$\rightarrow \int_0^2 \int_0^{x^4} e^{x^5}\, dy\, dx = \int_0^2 \left[y e^{x^5} \right]_0^{x^4} dx$$

$$= \int_0^2 x^4 e^{x^5}\, dx = \left[\frac{1}{5} e^{x^5} \right]_0^2$$

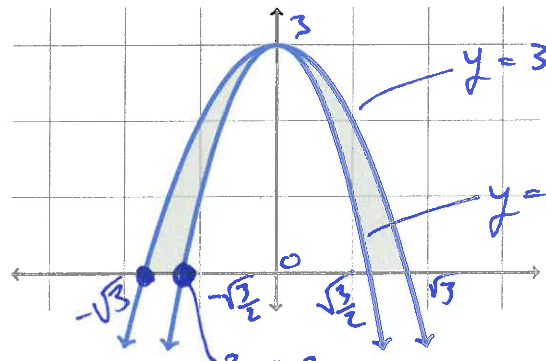
$$= \frac{1}{5} e^{32} - \frac{1}{5} e^0$$

$$= \boxed{\frac{1}{5}\left(e^{32} - 1\right)}$$

(d) Calculate $\int_2^3 \int_1^4 x\, dy\, dx$.

$$= \int_2^3 \left[xy \right]_1^4 dx = \int_2^3 3x\, dx = \frac{3}{2} x^2 \Big]_2^3 = \frac{27}{2} - \frac{12}{2} = \boxed{\frac{15}{2}}$$

(e) Describe the shaded region below in Cartesian coordinates. The outer function is $y = 3 - x^2$ and the inner function is $y = 3 - 2x^2$.



$y = 3 - x^2 \Rightarrow x = \pm\sqrt{3-y}$

$y = 3 - 2x^2 \Rightarrow x = \pm\sqrt{\frac{3-y}{2}}$

$3 - 2x^2 = 0$

$\Rightarrow x = \pm\sqrt{\frac{3}{2}}$

Easier way:

$$\left\{ \begin{array}{l} y \text{ in } [0,3] \\ x \text{ in } \left[-\sqrt{3-y}, -\sqrt{\frac{3-y}{2}}\right] \end{array} \right\} + \left\{ \begin{array}{l} y \text{ in } [0,3] \\ x \text{ in } \left[\sqrt{\frac{3-y}{2}}, \sqrt{3-y}\right] \end{array} \right\}$$

left half right half

Harder way:

$$\left\{ \begin{array}{l} x \text{ in } \left[-\sqrt{3}, -\sqrt{\frac{3}{2}}\right] \\ y \text{ in } [0, 3-x^2] \end{array} \right\}$$

$$+ \left\{ \begin{array}{l} x \text{ in } \left[-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right] \\ y \text{ in } [3-2x^2, 3-x^2] \end{array} \right\}$$

$$+ \left\{ \begin{array}{l} x \text{ in } \left[\sqrt{\frac{3}{2}}, \sqrt{3}\right] \\ y \text{ in } [0, 3-x^2] \end{array} \right\}$$

(f) Convert the following integral to spherical coordinates. *Do not evaluate.*

$$\underbrace{\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}}_{\text{circle in xy-plane of radius 2}}\int_{2-\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} (xy+z)\,dz\,dy\,dx$$

← sphere centered at (0,0) w/ radius 2

↖ cone facing down with peak at 2

→ θ in $[0, 2\pi]$

φ in $[0, \pi/2]$

bottom: $z = 2 - \sqrt{x^2+y^2}$

$\Rightarrow \rho\cos(\varphi) = 2 - \rho\sin(\varphi)$

$\Rightarrow \rho = \dfrac{2}{\cos(\varphi)+\sin(\varphi)}$

top: $z = \sqrt{4-x^2-y^2}$

$\Rightarrow \rho = 2$

$xy+z \Rightarrow \rho^2\sin^2(\varphi)\sin(\theta)\cos(\theta) + \rho\cos(\varphi)$

$$\int_0^{2\pi}\int_0^{\pi/2}\int_{\frac{2}{\cos(\varphi)+\sin(\varphi)}}^{2} \left(\rho^2\sin^2(\varphi)\sin(\theta)\cos(\theta) + \rho\cos(\varphi)\right)\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta$$

$= \rho^3\sin(\varphi)\left(\rho\sin^2(\varphi)\sin(\theta)\cos(\theta) + \cos(\varphi)\right)$

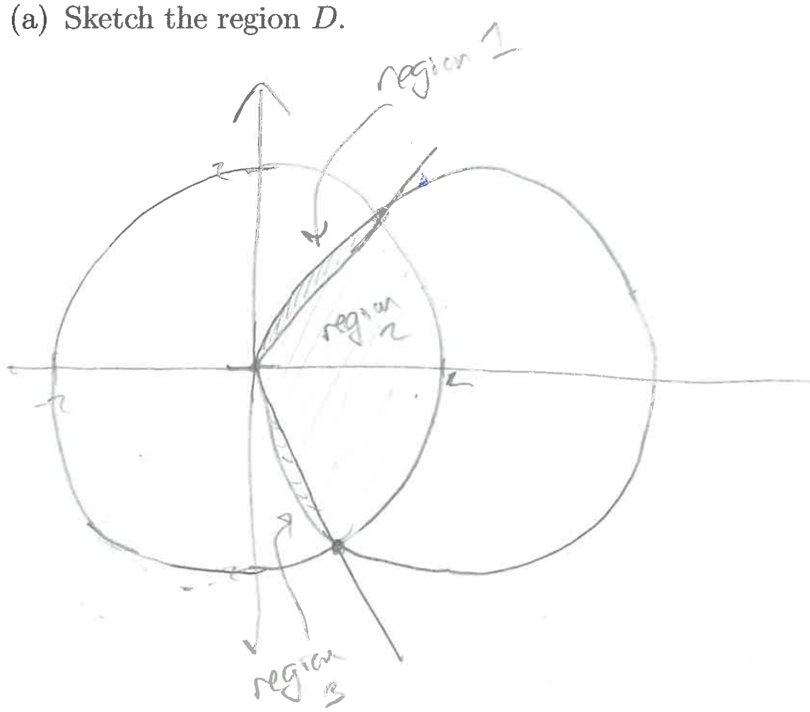
Section 3: Long Answer.

You must show all work to receive credit. If you need more space you may use the back of the page. You must clearly indicate on the front of the page that there is more work on the back of the page. Please work neatly and circle your final answer.

3. (8) Let C_1 be the circle centered at the origin with radius 2 and let C_2 be the circle centered at $(2,0)$ with radius 2. Let D be the region that lies within both circles.

2pts

(a) Sketch the region D.



3pts

(b) Describe the region D in polar coordinates.

Intersection points. $C_1: r=2$

$C_2: r=4\cos(\theta)$

$2=4\cos(\theta) \Rightarrow \cos(\theta)=\frac{1}{2}$

$\Rightarrow \theta=\pm\frac{\pi}{3}$

region 2: $\left\{\begin{array}{l}\theta \text{ in } [-\frac{\pi}{3},\frac{\pi}{3}] \\ r \text{ in } [0,2]\end{array}\right\}$

Don't forget regions 1 and 3!

region 1: $\left\{\begin{array}{l}\theta \text{ in } [\frac{\pi}{3},\frac{\pi}{2}] \\ r \text{ in } [0, 4\cos(\theta)]\end{array}\right\}$

region 3: $\left\{\begin{array}{l}\theta \text{ in } [-\frac{\pi}{2},-\frac{\pi}{3}] \\ r \text{ in } [0,4\cos(\theta)]\end{array}\right\}$

3pts

(c) Calculate $\iint_D \sqrt{x^2+y^2}\,dA$.

The integrand is r. Observe that the integrals over region 1 and 3 are equal because the areas are the same and the integrand $f(r,\theta)=r$ takes the same values.

So, $$\int_{\pi/3}^{\pi/2}\int_0^{4\cos(\theta)} r^2\,dr\,d\theta = \int_{\pi/3}^{\pi/2}\left[\frac{r^3}{3}\right]_0^{4\cos\theta} d\theta = \int_{\pi/3}^{\pi/2} \frac{64}{3}\cos^3(\theta)\,d\theta$$

$$\cos^3(\theta) = \cos(\theta)\left(1-\sin^2(\theta)\right) = \cos(\theta) - \cos(\theta)\sin^2(\theta)$$

$$= \frac{64}{3}\left[\sin(\theta) - \frac{\sin^3(\theta)}{3}\right]_{\pi/3}^{\pi/2} = \frac{64}{3}\left[\left(1-\frac{1}{3}\right) - \left(\frac{\sqrt{3}}{2} - \frac{\left(\frac{\sqrt{3}}{2}\right)^3}{3}\right)\right] \quad \left(\frac{\sqrt{3}}{2}\right)^3 = \frac{\sqrt{3}}{8}$$

$$= \frac{64}{3}\left(\frac{2}{3} - \frac{3\sqrt{3}}{8}\right) = \frac{128}{9} - 8\sqrt{3}$$

Region 2: $$\int_{-\pi/3}^{\pi/3}\int_0^2 r^2\,dr\,d\theta = \int_{-\pi/3}^{\pi/3}\left[\frac{r^3}{3}\right]_0^2 d\theta = \int_{-\pi/3}^{\pi/3} \frac{8}{3}\,d\theta = \frac{2\pi}{3}\cdot\frac{8}{3} = \frac{16\pi}{9}$$

Answer: $$\frac{16\pi}{9} + 2\left(\frac{128}{9} - 8\sqrt{3}\right)$$

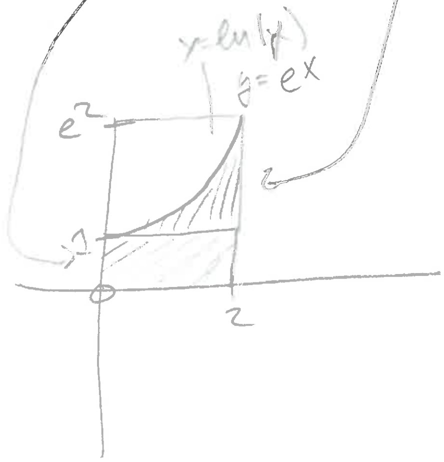
$$= \frac{16\pi}{9} + \frac{256}{9} - 16\sqrt{3}$$

4. (8) Consider the integral

$$I = \int_0^1 \int_0^2 (y+3)\,dx\,dy + \int_1^{e^2} \int_{\ln(y)}^2 (y+3)\,dx\,dy.$$

(a) Sketch the regions of integration of the two integrals on the same axis.

2pts



(b) Express I as a single integral with the order of integration reversed.

3pts

$$\int_{x=0}^{2} \int_{y=0}^{e^x} (y+3)\,dy\,dx$$

(c) Evaluate the integral that you found in part (b).

3 pts

$(e^x)^2 = e^{2x}$, not $e^{(x^2)}$

$$= \int_{x=0}^{2} \left[\frac{y^2}{2} + 3y\right]_0^{e^x} dx = \int_{x=0}^{2} \left(\frac{(e^x)^2}{2} + 3e^x\right) dx = \int_{x=0}^{2} \left(\frac{e^{2x}}{2} + 3e^x\right) dx$$

$$= \left[\frac{1}{4}e^{2x} + 3e^x\right]_0^2 = \frac{1}{4}e^4 + 3e^2 - \left(\frac{1}{4}e^0 + 3e^0\right)$$

$$\boxed{= \frac{1}{4}e^4 + 3e^2 - \frac{13}{4}}$$

5. (8)

centered at the origin

4 pts

(a) Find the average distance of a point on a disk of radius 4 from the origin. (*Hint:* Let $d(x,y) = \sqrt{x^2+y^2}$. Then, $d(x,y)$ is the distance of a point from the origin.)

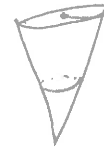
$$d(x,y) = \sqrt{x^2+y^2}$$

Disk: $\left\{ \begin{matrix} \theta \text{ in } [0, 2\pi] \\ r \text{ in } [0,4] \end{matrix} \right\}$ Area $= \pi(4)^2 = 16\pi$.

$$\int_0^{2\pi}\int_0^4 r^2\,dr\,d\theta = \int_0^{2\pi} \left[\frac{r^3}{3}\right]_0^4 d\theta = \int_0^{2\pi} \frac{64}{3}\,d\theta = \frac{128\pi}{3}.$$

$$\text{Average distance} = \frac{\frac{128\pi}{3}}{16\pi} = \boxed{\frac{8}{3}}$$

4 pts (b) Find the average height (z-value) of a point in the solid cone bounded by $z \geq \sqrt{x^2+y^2}$ and $z \leq 2$.



$h(x,y,z) = z$

Cylindrical Region: θ in $[0, 2\pi]$

r in $[0, 2]$

z in $[r, 2]$

Volume $= \frac{1}{3} \cdot (\pi \cdot (2)^2) \cdot 2 = \frac{8\pi}{3}$

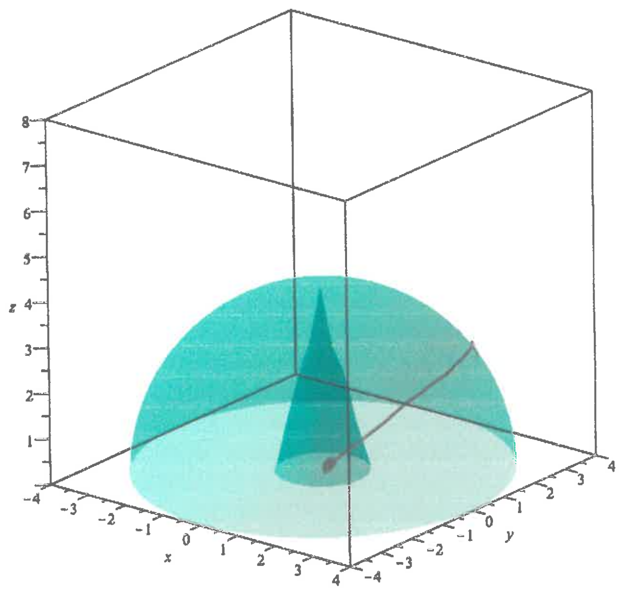
$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \int_{z=r}^{z=2} z r \, dz \, dr \, d\theta = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \left[\frac{z^2}{2} r\right]_r^2 dr \, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \left(2r - \frac{r^3}{2}\right) dr \, d\theta = \int_{\theta=0}^{2\pi} \left[r^2 - \frac{r^4}{8}\right]_0^2 d\theta$$

$$= \int_0^{2\pi} (4-2) \, d\theta = 4\pi.$$

Avg. height is $\frac{4\pi}{8\pi/3} = \frac{4\pi \cdot 3}{{}_2 8\pi} = \boxed{\frac{3}{2}}$

6. (8) Consider the region W that lies above $z = 0$, above the upside-down cone $z = 4 - 4\sqrt{x^2+y^2}$, and inside the sphere $x^2 + y^2 + z^2 = 16$. See the picture below. Express the volume of W (that is, the integral of $f(x, y, z) = 1$) as requested below.



(a) Express, but *do not evaluate*, the volume of W as an integral or sum of integrals with spherical coordinates.

3pts

Cone: $z = 4 - 4\sqrt{x^2+y^2}$ Sphere: $\rho = 4$

$\rho\cos(\varphi) = 4 - 4\rho\sin(\varphi)$

$\rho = \dfrac{4}{\cos(\varphi) + 4\sin(\varphi)}$

bottom = cone, top = sphere

$$\int_0^{2\pi}\int_0^{\pi/2}\int_{\frac{4}{\cos(\varphi)+4\sin(\varphi)}}^{4} \rho^2\sin(\varphi)\, d\rho\, d\varphi\, d\theta$$

3pts

(b) Express, but *do not evaluate*, the volume of W as an integral or sum of integrals with cylindrical coordinates.

Two regions of projection onto xy-plane

under cone: $\{\theta \text{ in } [0, 2\pi]\}$, $\{r \text{ in } [0, 1]\}$

outside cone: $\{\theta \text{ in } [0, 2\pi]\}$, $\{r \text{ in } [1, 4]\}$

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=4-4r}^{z=\sqrt{16-r^2}} r \, dz\, dr\, d\theta + \int_{\theta=0}^{2\pi} \int_{r=1}^{4} \int_{z=0}^{z=\sqrt{16-r^2}} r \, dz\, dr\, d\theta$$

2pts

(c) Explain briefly why it would be quite laborious to express the volume of W as an integral or sum of integrals with Cartesian coordinates.

Like (b), we need to do the circle with radius 1 separately, leaving the outer part, which would need to split into several regions to do with Cartesian coordinates