NAME : Key

Math 13

Midterm 2

November 1, 2016

Prof. Pantone

INSTRUCTIONS: This is a closed book exam and no notes are allowed. You are not to provide or receive help from any outside source during the exam except that you may ask the instructor for clarification of a problem. You have 120 minutes and you should attempt all problems.

- Print your name in the space provided.
- Calculators or other computing devices are not allowed.
- Except when indicated, you must show all work and give justification for your answer. **A correct answer with incorrect work will be considered wrong.**

All work on this exam should be completed in accordance with the Dartmouth Academic Honor Principle.

Helpful Hints:

- Use your time wisely. If you're worried about having enough time, start with the problems that are worth more points.
- Work carefully and methodically to prevent arithmetic errors.
- Work neatly so you have enough room.
- Don't Panic!

Problem	Points	Score
1	6	
2	12	
3	8	
4	8	
5	8	
6	8	
Total	50	

Section 1: True/False.

1. (6) Choose the correct answer. *No justification is required for your answers. No partial credit will be awarded.*

(a) Linear changes of coordinates map rectangles to rectangles.

Parallelograms to parallelograms

True False

(b) The flux of a vector field over a region is defined to be the amount of material flowing out of the region.

True False

Both accepted. I meant the answer to be true, but since I left out the word "net" it could be misunderstood.

(c) The circulation of a vector field over a region is the amount of material flowing around the boundary of the region in the ~~clockwise~~ direction.

counterclockwise

True False

(d) Integration in vector fields is path independent: if $\boldsymbol{F}$ is a vector field and C_1 and C_2 are two curves with the same starting point P and ending point Q, then the line integrals of $\boldsymbol{F}$ over C_1 and C_2 are equal.

Only integration in CVFs is path independent

True False

(e) Notationally, $\text{curl}(\boldsymbol{F}) = \nabla \times \boldsymbol{F}$ and $\text{div}(\boldsymbol{F}) = \nabla \cdot \boldsymbol{F}$

True False

(f) Let $\boldsymbol{F}$ be a vector field and let C be a closed curve. If $\displaystyle\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0$, then $\boldsymbol{F}$ is a conservative vector field.

True False

$[F \text{ is a CVF}] \Rightarrow [\oint_C \vec{F} = 0 \text{ for all } C]$

But $[\oint_C \vec{F} = 0 \text{ for one } C] \not\Rightarrow [\vec{F} \text{ is a CVF}]$

Section 2: Short Answer.

2. (12) *Circle your answer. No justification is required. No partial credit will be awarded.*

(a) Find the area enclosed by the ellipse $\langle \cos(t) + \sin(t),\, \sin(t)\rangle$ for $0 \le t \le 2\pi$. (*Hint:* Use Green's Theorem with $\boldsymbol{F} = \left\langle -\frac{y}{2}, \frac{x}{2} \right\rangle$.)

Let D be the ellipse

By GT,

$$\text{Area}(D) = \iint_D 1\,dA = \oint_C \left\langle -\tfrac{y}{2}, \tfrac{x}{2} \right\rangle d\vec{r}.$$

$$\vec{r}(t) = \langle \cos(t) + \sin(t), \sin(t) \rangle \Rightarrow \vec{r}\,'(t) = \langle \cos(t) - \sin(t), \cos(t) \rangle.$$

$$\oint_C \left\langle -\tfrac{y}{2}, \tfrac{x}{2} \right\rangle d\vec{r} = \int_0^{2\pi} \left\langle -\frac{\sin(t)}{2}, \frac{\cos(t)+\sin(t)}{2} \right\rangle \cdot \langle \cos(t) - \sin(t), \cos(t) \rangle\, dt$$

$$= \int_0^{2\pi} \left(\cancel{-\frac{\cos(t)\sin(t)}{2}} + \frac{\sin^2(t)}{2} + \frac{\cos^2(t)}{2} + \cancel{\frac{\sin(t)\cos(t)}{2}} \right) dt = \int_0^{2\pi} \frac{1}{2}\, dt = \boxed{\pi}$$

(b) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the vector field $\mathbf{F}(x, y, z) = \langle y + z, x + z, x + y \rangle$ and the curve C parametrized as $\mathbf{r}(t) = \langle e^{t^3}, e^t, e^{2t} \rangle$ with $0 \le t \le 2$.

$\vec{F}$ is a CVF with potential function

$f(x,y,z) = xy + xz + yz$.

(We saw the same $\vec{F}$ on a homework assignment.)

So, by the FTCVF: $\int_C \vec{F}\, d\vec{r} = f(\vec{r}(2)) - f(\vec{r}(0))$

$= f(e^8, e^2, e^4) - f(1,1,1)$

$= (e^8)(e^2) + (e^8)(e^4) + (e^2)(e^4) - 3$

$= \boxed{e^{10} + e^{12} + e^6 - 3}$

(c) Find a potential function for the vector field

$$\mathbf{F} = \langle e^y + yze^x,\ xe^y + e^x z,\ e^x y\rangle$$

$$f(x,y,z) = \boxed{xe^y + yze^x}$$

Verification: $\nabla f = \langle e^y + yze^x,\ xe^y + ze^x,\ ye^x \rangle$

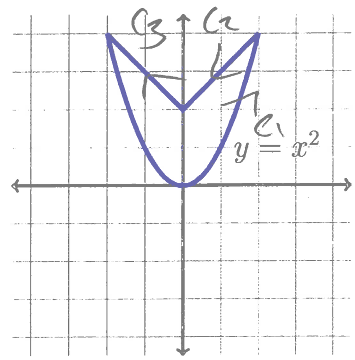
(d) Compute the curl of $\mathbf{F} = \langle x^3y^2z,\ x^2z,\ x^2y\rangle$.

$$\text{Curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3y^2z & x^2z & x^2y \end{vmatrix}$$

$$= \langle x^2 - x^2,\ -(2xy - x^3y^2),\ 2xz - 2x^3yz \rangle$$

$$= \langle 0,\ x^3y^2 - 2xy,\ 2xz - 2x^3yz \rangle$$

(e) Provide a parametrization of the curve below, oriented counterclockwise.



$C_1 = \langle t, t^2 \rangle \quad -2 \le t \le 2$

$C_2 = \langle 2-t, 4-t \rangle \quad 0 \le t \le 2$

$C_3 = \langle -t, 2+t \rangle \quad 0 \le t \le 2$

(f) Let G be the transformation $G(u, v) = (3u - v, u + v)$. (In other words, $x(u, v) = 3u - v$ and $y(u, v) = u + v$.) Compute $\text{Jac}(G)$.

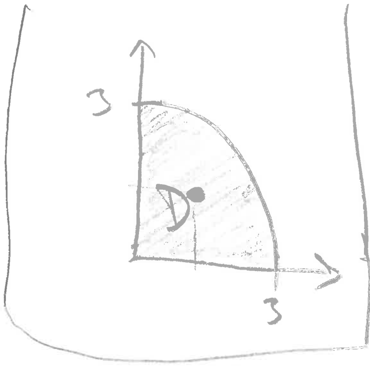
$x(u,v) = 3u - v$
$y(u,v) = u + v$

$$\text{Jac}(G) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} = 3 - (-1) = \boxed{4}$$

Section 3: Long Answer.

You must show all work to receive credit. If you need more space you may use the back of the page. You must clearly indicate on the front of the page that there is more work on the back of the page. Please work neatly and circle your final answer.

3. (8) Find the center of mass of the region in the xy-plane that consists of the upper-right quarter of the circle $x^2 + y^2 = 9$ (i.e., the part of the circle lying in the first quadrant), given mass density $\delta(x, y) = \sqrt{x^2 + y^2}$.



$$\delta(x,y) = \sqrt{x^2+y^2} = \sqrt{r^2\cos^2(\theta)+r^2\sin^2(\theta)} = \sqrt{r^2} = r$$

$$\text{Mass} = \iint_D \delta(x,y)\,dA$$

correction factor

$$= \int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=3} r\cdot r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=\pi/2}\left[\frac{r^3}{3}\right]_0^3 d\theta = \int_{\theta=0}^{\theta=\pi/2} 9\,d\theta = \frac{9\pi}{2}$$

$$M_y = \iint_D x\,\delta(x,y)\,dA = \int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=3} r^3\cos(\theta)\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2}\left[\frac{r^4}{4}\cos(\theta)\right]_0^3 d\theta = \int_0^{\pi/2}\frac{81}{4}\cos(\theta)\,d\theta$$

$$= \left[\frac{81}{4}\sin(\theta)\right]_0^{\pi/2} = \frac{81}{4}$$

$$M_x = \iint_D y\,\delta(x,y)\,dA = \int_{\theta=0}^{\pi/2}\int_{r=0}^{3} r^3\sin(\theta)\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2}\left[\frac{r^4}{4}\sin(\theta)\right]_0^3 d\theta = \int_{\theta=0}^{\pi/2}\frac{81}{4}\sin(\theta)\,d\theta$$

$$= \left[-\frac{81}{4}\cos(\theta)\right]_0^{\pi/2} = \frac{81}{4}$$

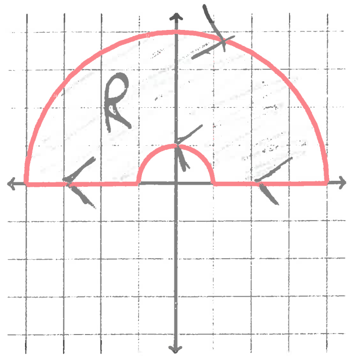
$$\text{COM} = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$$

$$= \left(\frac{\cancel{81}^{\,9}}{\cancel{4}_{\,2}}\cdot\frac{\cancel{2}}{\cancel{9\pi}}, \frac{\cancel{81}^{\,9}}{\cancel{4}_{\,2}}\cdot\frac{\cancel{2}}{\cancel{9\pi}}\right)$$

$$\boxed{= \left(\frac{9}{2\pi}, \frac{9}{2\pi}\right)}$$

$$\approx (1.43, 1.43)$$

4. (8) Evaluate $\oint_C \langle e^{\arctan(x)} + 2y^2, \cos(y) + x\rangle \cdot d\mathbf{r}$, where C is the curve below, oriented clockwise. Both of the curves below are semicircles.



When you see something like $e^{\arctan(x)}$, you should think of Green's Theorem! Green's Theorem says

$$\oint_{-C} \vec{F}\,d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \iint_R (1-4y)\,dA$$

since our C is clockwise

convert to polar: $\displaystyle\int_{\theta=0}^{\theta=\pi}\int_{r=1}^{r=4}(1-4r\sin(\theta))\,r\,dr\,d\theta$

$$= \int_{\theta=0}^{\theta=\pi}\int_{r=1}^{r=4}(r-4r^2\sin(\theta))\,dr\,d\theta = \int_{\theta=0}^{\theta=\pi}\left[\frac{r^2}{2}-\frac{4r^3}{3}\sin(\theta)\right]_1^4 d\theta$$

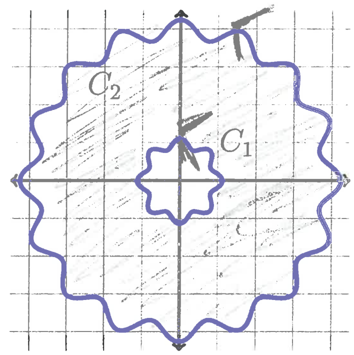
$\frac{256}{3} - \frac{4}{3} = \frac{252}{3}$ $\quad 84 \quad 3\overline{)252}$ $\quad 24 \quad 12$

$$= \int_{\theta=0}^{\theta=\pi}\left(\left(8-\frac{256}{3}\sin(\theta)\right)-\left(\frac{1}{2}-\frac{4}{3}\sin(\theta)\right)\right)d\theta = \int_{\theta=0}^{\theta=\pi}\left(\frac{15}{2}-84\sin(\theta)\right)d\theta$$

$$= \left[\frac{15}{2}\theta + 84\cos(\theta)\right]_0^{\pi} = \frac{15}{2}\pi - 84 - 84 = \frac{15}{2}\pi - 168$$

So, $\displaystyle\oint_C \vec{F}\,d\vec{r} = -\oint_{-C}\vec{F}\,d\vec{r} = \boxed{168 - \frac{15}{2}\pi}$

5. (8) Consider the vector field $\boldsymbol{F} = \langle F_1, F_2 \rangle = \left\langle -y - \frac{y}{x^2+y^2},\, x + \frac{x}{x^2+y^2} \right\rangle$ and the curves below, both oriented counterclockwise.



The strange curve tells us we shouldn't try to integrate!

Suppose you know that

$$\oint_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} = 3 + \pi.$$

and that the *area* of the region *between* the curves C_1 and C_2 is 7. Use Green's Theorem to calculate

$$\oint_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

(*Hint:* $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 2$. You should be able to answer this question without doing any actual integration.)

By GT: $\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = -\oint_{C_1} \vec{F} d\vec{r} + \oint_{C_2} \vec{F} dr$

$\iint_R 2\,dA = 2\,\text{Area}(R) = 14$; $-\oint_{C_1} \vec{F} d\vec{r} = -(3+\pi)$; $\oint_{C_2} \vec{F} dr =$ unknown.

So, $14 = -3 - \pi + [\text{unknown}]$

Thus, $\oint_{C_2} \vec{F} d\vec{r} = \boxed{17+\pi}$

6. (8) Consider the helix parametrized by $\boldsymbol{r}(t) = \langle \cos(t),\, \sin(t),\, t \rangle$ for $0 \le t \le 4\pi$. Calculate

$$\int_C (1 + x + z)\, ds.$$

$$\vec{r}\,'(t) = \langle -\sin(t), \cos(t), 1 \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{2}$$

$$\int_C (1+x+z)\,ds = \int_0^{4\pi} (1 + \cos(t) + t)\sqrt{2}\,dt$$

$$= \sqrt{2}\left[t + \frac{t^2}{2} + \sin(t)\right]_0^{4\pi}$$

$$= \sqrt{2}(4\pi + 8\pi^2 + 0) - \sqrt{2}(0 + 0 + 0)$$

$$= 4\sqrt{2}\,\pi + 8\sqrt{2}\,\pi^2$$