Math 13 Homework 3 Solutions

Q2: 7pts
Q3: 7pts
Q4: 6pts

Q1 moved to homework 4.

(no page 2)

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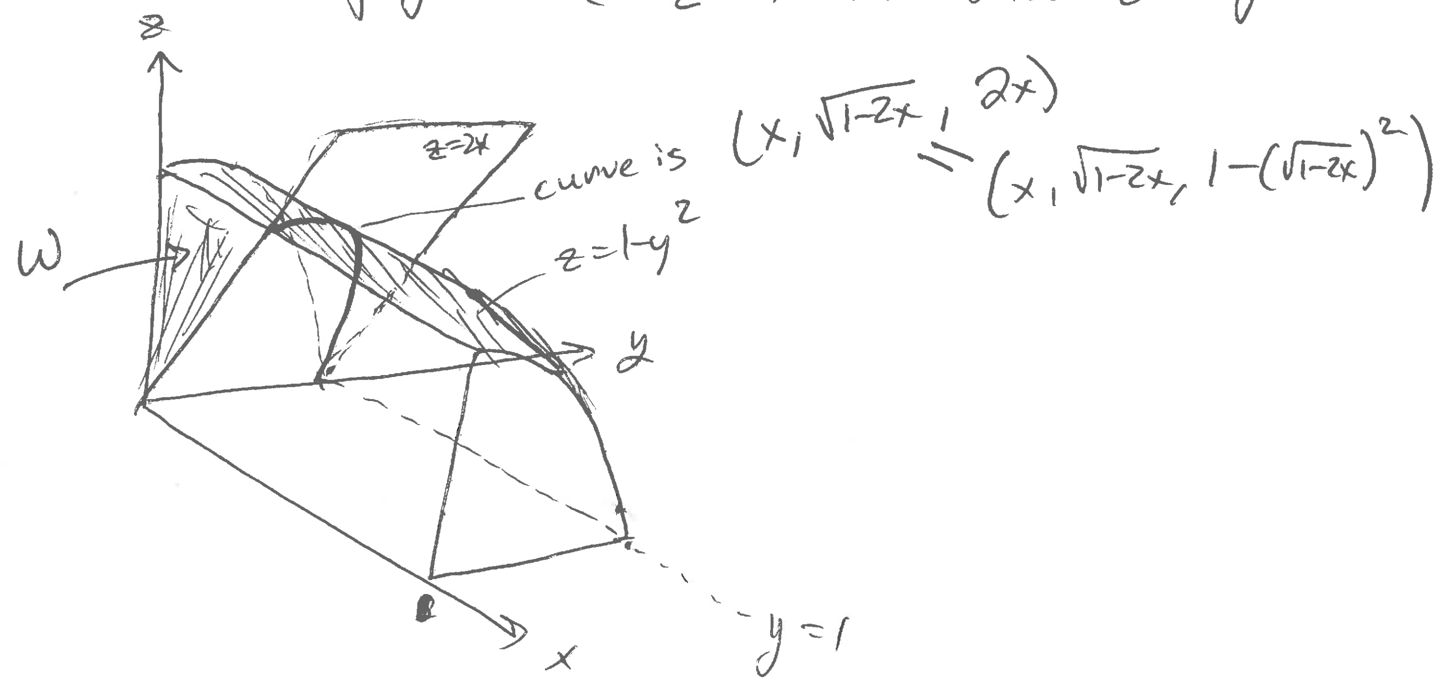
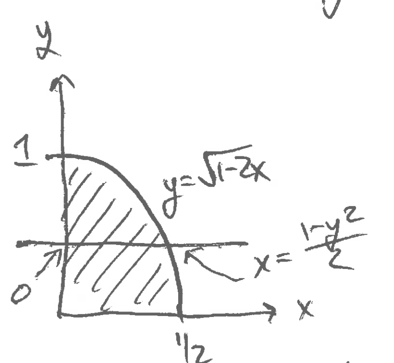
2) Find the average value of the function $f(x,y,z)=y$ over the region W in the first octant lying above $z=2x$ and below $z=1-y^2$



Region is z-simple. Shadow on the xy-plane is...

first we need the intersection curve of the surfaces.

$$2x=1-y^2 \Rightarrow y^2=1-2x \Rightarrow y=\sqrt{1-2x}$$



First we need the volume of the solid.

$$\int_{y=0}^{y=1} \int$$

$$\text{Volume}(W) = \int_{y=0}^{y=1} \int_{x=0}^{x=\frac{1-y^2}{2}} \int_{z=2x}^{z=1-y^2} 1 \, dz \, dx \, dy$$

$$= \int_{y=0}^{y=1} \int_{x=0}^{x=\frac{1-y^2}{2}} \left[z \right]_{z=2x}^{z=1-y^2} dx \, dy = \int_{y=0}^{y=1} \int_{x=0}^{x=\frac{1-y^2}{2}} (1-y^2-2x) \, dx \, dy$$

$$= \int_{y=0}^{y=1} \left[x - x^2 - xy^2 \right]_{x=0}^{x=\frac{1-y^2}{2}} dy = \int_{y=0}^{y=1} \left(\left(\frac{1-y^2}{2}\right) - \left(\frac{1-y^2}{2}\right)^2 - \left(\frac{1-y^2}{2}\right) y^2 \right) dy$$

$$= \int_{y=0}^{y=1} \left(\frac{1}{2} \cancel{- \frac{y^2}{2}} - \frac{1}{4} \cancel{+ \frac{y^2}{2}} \cancel{- \frac{y^4}{4}} \cancel{- \frac{y^2}{2}} \cancel{+ \frac{y^4}{2}} \right) dy$$

$$= \int_{y=0}^{y=1} \left(\frac{y^4}{4} - \frac{y^2}{2} + \frac{1}{4} \right) dy = \left[\frac{y^5}{20} - \frac{y^3}{6} + \frac{y}{4} \right]_{y=0}^{y=1}$$

$$= \frac{1}{20} - \frac{1}{6} + \frac{1}{4} = \frac{3}{60} - \frac{10}{60} + \frac{15}{60} = \frac{8}{60} = \boxed{\frac{2}{15}}$$

Next we need $\iiint_W y\, dV$

But there's a shortcut! This integral is:

$$\int_{y=0}^{y=1}\int_{x=0}^{x=\frac{1-y^2}{2}}\int_{z=2x}^{z=1-y^2} y\, dz\,dx\,dy = \int_{y=0}^{y=1}\left(y\cdot\underbrace{\left(\int_{x=0}^{x=\frac{1-y^2}{2}}\int_{z=2x}^{z=1-y^2} 1\, dz\,dx\right)}\right)dy$$

and we already found in the last step that this equals $\frac{y^4}{4}-\frac{y^2}{2}+\frac{1}{4}$. So,

$$\iiint_W y\, dV = \int_{y=0}^{y=1} y\left(\frac{y^4}{4}-\frac{y^2}{2}+\frac{1}{4}\right)dy =$$

$$\left[\frac{y^6}{24}-\frac{y^4}{8}+\frac{y^2}{8}\right]_{y=0}^{y=1} = \frac{1}{24}-\frac{1}{8}+\frac{1}{8}=\frac{1}{24}.$$

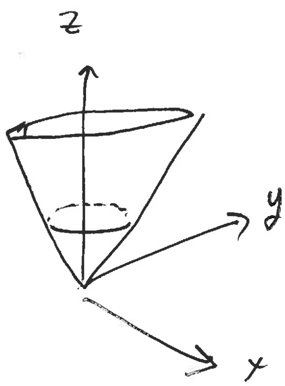
Therefore, the average y-value is

$$\frac{1/24}{2/15} = \frac{1}{24}\cdot\frac{15}{2}=\frac{15}{48}=\boxed{\frac{5}{16}}.$$

3) Express, but <u>do not evaluate</u>, the integral of $x+2y+z$ over the region in the first octant above $z=\sqrt{x^2+y^2}$ and below the surface $z=\sqrt{4-x^2-y^2}$ in

a) Cartesian coordinates

We need a picture! $\sqrt{x^2+y^2}$ is the distance from $(0,0)$ to (x,y). So after a little thinking, we see $z=\sqrt{x^2+y^2}$ is a cone with the point at $(0,0)$.

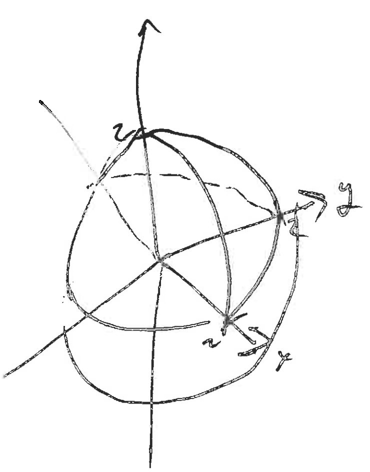


What is $z=\sqrt{4-x^2-y^2}$?

$\Rightarrow z^2=4-x^2-y^2$

$\Rightarrow x^2+y^2+z^2=4$

$\Rightarrow$ sphere with center $(0,0,0)$ and radius 2.



Where do they intersect?

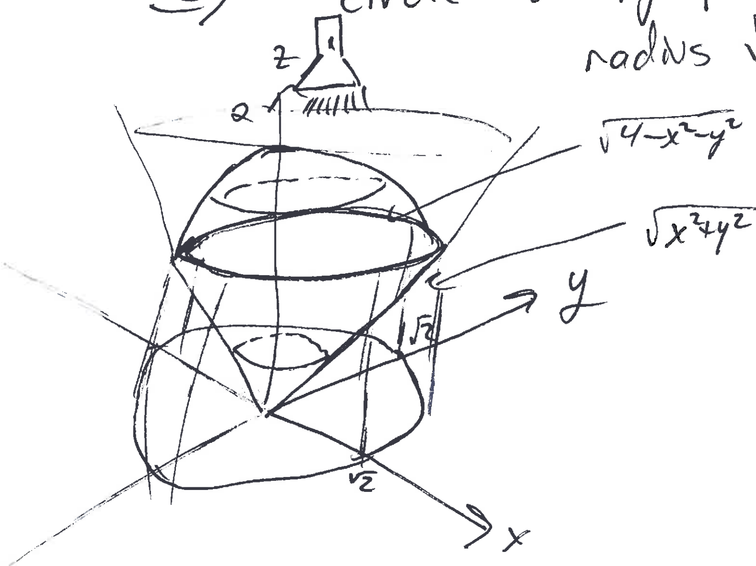
$$\sqrt{x^2+y^2} = \sqrt{4-x^2-y^2}$$

$$\Rightarrow \quad x^2+y^2 = 4-x^2-y^2$$

$$\Rightarrow \quad 2x^2+2y^2 = 4$$

$$\Rightarrow \quad x^2+y^2 = 2$$

$\Rightarrow$ circle in xy-plane with radius $\sqrt{2}$



first quadrant of circle in xy-plane with radius $\sqrt{2}$ is

x in $[\cancel{-\sqrt{2}}\ 0, \sqrt{2}]$

y in $[\cancel{-\sqrt{2-x^2}}\ 0, \sqrt{2-x^2}]$

So,

$$\int_{x=\cancel{-\sqrt{2}}\ 0}^{x=\sqrt{2}} \int_{y=\cancel{-\sqrt{2-x^2}}\ 0}^{y=\sqrt{2-x^2}} \int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{4-x^2-y^2}} (x+2y+z)\,dz\,dy\,dx$$

b) Cylindrical

circle in first quadrant of xy-plane with radius $\sqrt{2}$:

θ in $[0, \cancel{2\pi}\ \pi/2]$

r in $[0, \sqrt{2}]$

$$z \text{ in } \left[\underbrace{\sqrt{x^2+y^2}}_{= r},\ \underbrace{\sqrt{4-x^2-y^2}}_{= \sqrt{4-r^2}}\right]$$

$\Longrightarrow$ z in $[r, \sqrt{4-r^2}]$.

$x+2y+z \longrightarrow r\cos\theta + 2r\sin(\theta) + z$

correction factor $\underline{r}$

Putting all together.

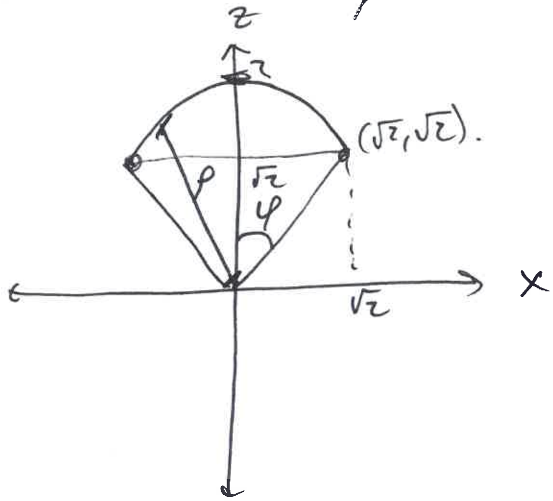
$$\int_{\theta=0}^{\theta=\cancel{2\pi}\ \pi/2} \int_{r=0}^{r=\sqrt{2}} \int_{z=r}^{z=\sqrt{4-r^2}} r\left(r\cos(\theta) + 2r\sin(\theta) + z\right) dz\, dr\, d\theta$$

c) Spherical

θ is easy, it goes all the way around the first quadrant

θ in $[0, \pi/2]$

What is the range for φ? Look at the xz-plane:



So φ starts at 0 (straight up) to $\pi/4$ $(0,0)$ to $(\sqrt{2}, \sqrt{2})$.

ρ range: From 0 to end of sphere (2).

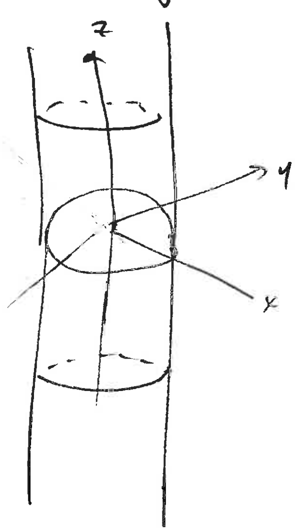
$$x + 2y + z \to \rho\sin(\varphi)\cos(\theta) + 2\rho\sin(\varphi)\sin(\theta) + \rho\cos(\varphi)$$

Correction factor: $\rho^2 \sin(\varphi)$

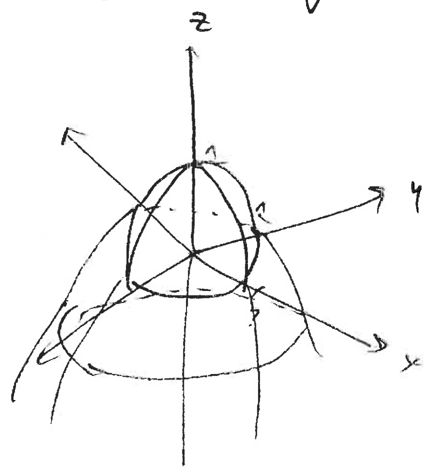
$$\to \int_{\theta=0}^{\pi/2} \int_{\varphi=0}^{\varphi=\pi/4} \int_{\rho=0}^{\rho=2} \rho^3 \sin(\varphi)\big(\sin(\varphi)\cos(\theta) + \sin(\varphi)\sin(\theta) + \cos(\varphi)\big)\, d\rho\, d\varphi\, d\theta$$

4) Let W be the solid region within $x^2+y^2=1$, above the paraboloid $z=1-x^2-y^2$ and below the plane $z=2$. Integrate $\sqrt{x^2+y^2}$ over this region.

$x^2+y^2=1$



$z=1-x^2-y^2$

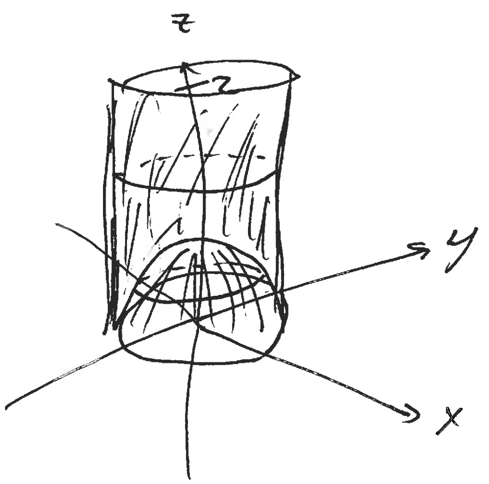


How do these two intersect?

(Think of it as: where does $z=1-x^2-y^2$ break out of the cylinder?)

When $x^2+y^2=1$ (meaning, you're on the boundary of the cylinder), the paraboloid is at height $z=1-x^2-y^2=1-\underbrace{(x^2+y^2)}_{=1}=0$.

So, the shape is the cylinder, capped at $z=2$, and with a hemisphere cut out of the bottom.



Cylindrical coordinates are best:

θ in $[0, 2\pi]$

r in $[0, 1]$

z in $[1-x^2-y^2, \cancel{\ } 2]$
$\quad\quad \| $
$[1-r^2, 2]$.

Integrating $\sqrt{x^2+y^2} \to r$

So, $$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=1-r^2}^{z=2} r^2 \, dz \, dr \, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \left[r^2 z \right]_{z=1-r^2}^{z=2} dr \, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \Big(\underbrace{2r^2 - \big(r^2(1-r^2)\big)}_{r^2+r^4} \Big) \, dr \, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} (r^2+r^4) \, dr \, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \left[\frac{r^3}{3} + \frac{r^5}{5} \right]_{r=0}^{r=1} d\theta \quad = \int_{\theta=0}^{\theta=2\pi} \left(\frac{1}{3} + \frac{1}{5} \right) d\theta$$

$$= \left[\frac{8}{15}\theta \right]_{\theta=0}^{\theta=2\pi} = \frac{16\pi}{15} .$$