Math 13 Homework #6

Due: November 10th, beginning of class

Show all of your work for full credit. Remember to sketch the region when asked. Simplify if there is an obvious way to do so, but some answers are ugly and do not need to be simplified.

1 Surface Integrals on Vector Fields

1. Compute $\int\int_S F \cdot dS$, where $F = \langle y, z, x\rangle$ and S is the portion of the plane $3x - 4y + z = 1$ for $0 \leq x \leq 1$, $0 \leq y \leq 1$ with upwards-pointing normals.
2. Compute $\int\int_S F \cdot dS$, where $F = \langle e^z, z, x\rangle$ and S is parametrized by $G(r, s) = (rs, r + s, r)$ for $0 \leq r \leq 1$, $0 \leq s \leq 1$, oriented by $T_r \times T_s$.
3. Compute $\int\int_S F \cdot dS$, where $F = \langle xz, yz, \frac{1}{z}\rangle$ and S is the disk of radius 3 centered at $x = 0, y = 0$ in the plane $z = 4$ with upwards-pointing normal vector.
4. Compute $\int\int_S F \cdot dS$, where $F = \langle y, z, 0\rangle$ and S is parametrized by $G(u, v) = (u^3 - v, u + v, v^2)$ for $0 \leq u \leq 2$, $0 \leq v \leq 3$ with downwards-pointing normal vectors.
5. Show that the flux of the vector field $F = \langle \frac{x}{\rho^3}, \frac{y}{\rho^3}, \frac{z}{\rho^3}\rangle$ through a sphere centered at the origin does not depend on the radius of the sphere.
6. Let $F = \langle a, b, c\rangle$ be any constant vector field. Calculate the flux of F through the sphere of radius R centered at the origin with outwards-facing normal vectors.
7. Compute $\int\int_S F \cdot dS$, where $F = \langle x, z, z\rangle$ and S is given by $y = 1 - x^2 - z^2$, $y \geq 0$, with normal vectors pointing in the positive y-direction.

2 Green's Theorem

1. Calculate $\oint_C xydx + ydy$ using Green's Theorem, where C is the unit circle oriented counterclockwise.

2. Calculate $\oint_C y^2 dx + x^2 dy$ using Green's Theorem, where C is the clockwise boundary of the unit square, $0 \leq x \leq 1$, $0 \leq y \leq 1$.

3. Calculate $\oint_C F \cdot dr$, where $F(x,y) = \langle x+y, x^2 - y\rangle$ and C is the counterclockwise boundary of the region enclosed by $y = x^2$ and $y = \sqrt{x}$ for $0 \leq x \leq 1$.

4. Calculate $\oint_C F \cdot dr$, where $F(x,y) = \langle x^2, x^2\rangle$ and C consists of the arcs $y = x^2$ and $y = x$ for $0 \leq x \leq 1$.

5. Let $F(x,y) = \langle 2xe^y, x + x^2 e^y\rangle$ and let C be the path which follows quarter circle from $(4,0)$ to $(0,4)$, then travels in a straight line to the origin, before moving in a straight line from the origin back to $(4,0)$.

   (a) Find a function f such that $F = G + \nabla f$, where $G(x,y) = \langle 0, x\rangle$.

   (b) Calculate the line integral of G along the line segment from $(0,0)$ to $(4,0)$, and evaluate the line integral of G along the line segment from $(0,4)$ to $(0,0)$.

   (c) Calculate the line integral of G along the quarter circle from $(4,0)$ to $(0,4)$.

   (d) Using your answers to the previous two parts, what is $\oint_C F \cdot dr$?

   (e) Use Green's Theorem to verify your answer to the previous part.

6. Use Green's Theorem to evaluate the integral $\int_C F \cdot dr$, where C is the path composed of line segments from the origin to $(2,2)$, then from $(2,2)$ to $(2,2)$, and finally from $(2,4)$ to $(0,6)$, and where $F(x,y) = \langle \sin(x) + y, 3x + y\rangle$. Hint: Add another line segment to close the path and apply Green's Theorem. From this, subtract the contribution of the added line segment.

7. Calculate the area between the x-axis and the path $r(t) = (t - \sin(t), 1 - \cos(t))$, $0 \leq t \leq 2\pi$ using Green's Theorem.

8. Calculate the area between the graph of $y = x^2$ and the x-axis for $0 \leq x \leq 2$ using Green's Theorem.

9. Suppose a lamina D has a simple, closed counterclockwise boundary C and has constant density 1. Use Green's Theorem to write the mass, moment along the x-axis, and the moment along the y-axis as line integrals over the boundary of C.

10. Use your answer to the previous question to find the center of mass of the semicircular lamina $x^2 + y^2 = 9$, $y \geq 0$, with constant density 1.