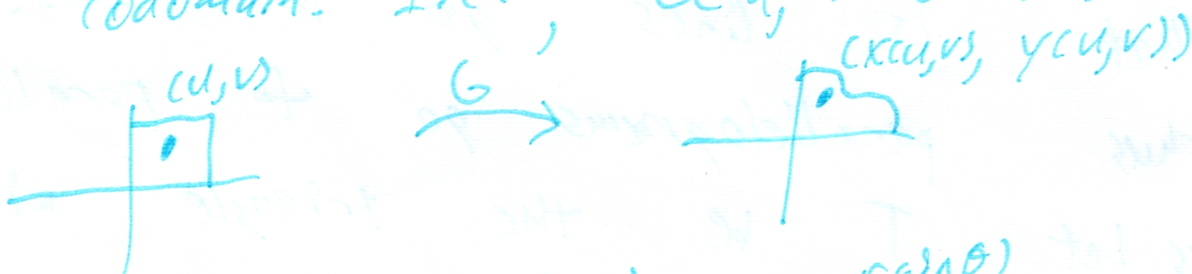
10/6 - Change of Variables

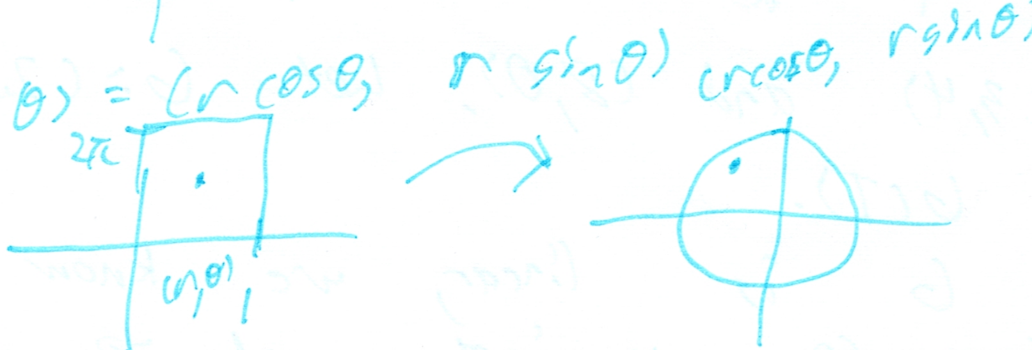
Change of variable is like u-substitution for multiple integrals.

A function $G: X \to Y$ (from X to Y) is sometimes called a map. X is the domain and Y is the codomain. Given a subset of X, U, $G(U) = \{G(x) : x \text{ in } U\}$. $G(U)$ is called the image of U under G, and $G(X)$ is the range of G. $G(x) = G(\{x\})$, which is called the ~~image~~ image of x.

For change of variables, we're given some D in $\mathbb{R}^2$, and thinking about maps $G: D \to \mathbb{R}^2$. u, v will be variables in the domain, and x, y will be variables in the codomain. I.e., $G(u,v) = (x(u,v), y(u,v))$.



Ex: $P(r, \theta) = (r\cos\theta, r\sin\theta)$



Ex: $P([r_1, r_2] \times P([\theta_1, \theta_2]) =$

$P(r=r_1) = (r_1 \cos\theta, r_1 \sin\theta)$

$P(r=r_2) = (r_2 \cos\theta, r_2 \sin\theta)$

$P(\theta=\theta_1) = (r\cos\theta_1, r\sin\theta_1)$

$P(\theta=\theta_2) = (r\cos\theta_2, r\sin\theta_2)$

The simplest types of maps are linear maps: $G(u,v) = (Au + Bv, Cu + Dv)$.

The nice property of linear maps is that $G(cu, cv) = cG(u,v)$ and

$$G(u_1 + u_2, v_1 + v_2) = G(u_1, v_1) + G(u_2, v_2).$$

That is, lines go to lines. (and thus parallelograms go to parallelograms.

Ex: Let T be the triangle with vertices $(2,1)$, $(3,4)$ and $(8,0)$. Let $G = (2u - v, u + v)$. Find $G(T)$.

Sol: As G is linear, we know lines go to lines, so we just need to look at

$G(2,1)$, $G(3,4)$, and $G(8,0)$

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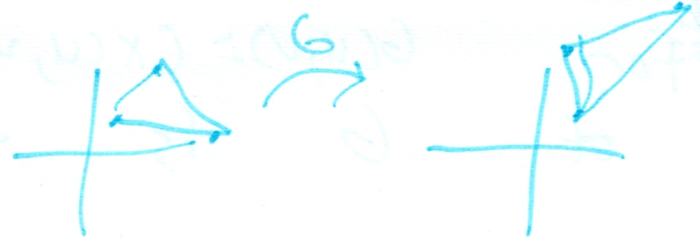
$(3,3) \qquad (2,7) \qquad\quad (16,8)$

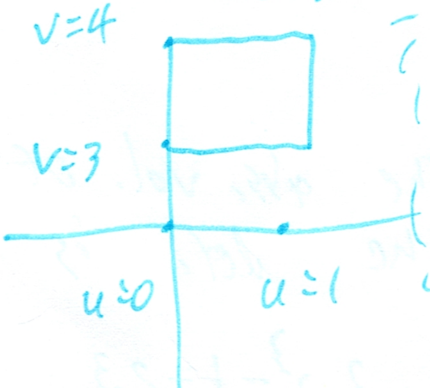


For non-linear maps, we look at the images of curves $v=f(u)$. Usually, the curves we pick are lines, but it depends on D

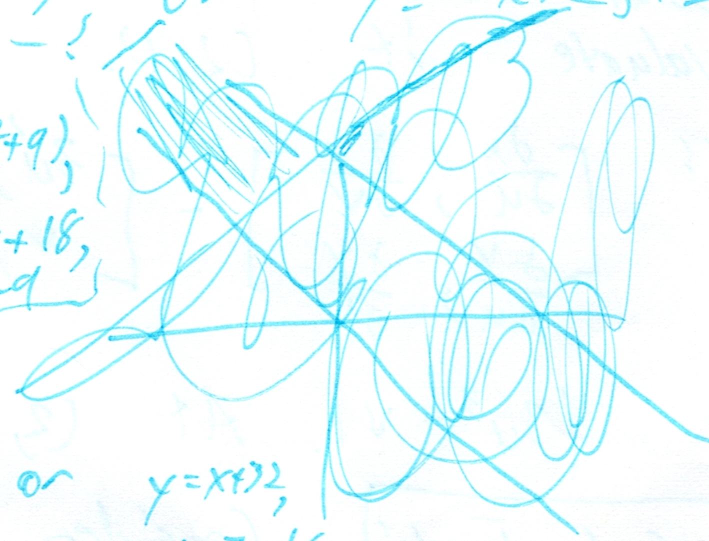
Ex: $G(u,v) = (u^2-v^2,\ u^2+v^2)$. Find the image of $[0,1]\times[3,4]$.

Sol: $G(0,v) = (-v^2,\ v^2)$, on $y=-x$, $x\le 0$

$G(1,v) = (1-v^2,\ 1+v^2)$, on $y=-x+2$, $x\le 1$



$G(u,3) = (u^2-9,\ u^2+9)$, or $y=x+18$, $x\ge -9$



$G(u,4) = (u^2-16,\ u^2+16)$, or $y=x+32$, $x\ge -16$

Stopped here on Wednesday