*For $G(u,v) = (x(u,v), y(u,v))$, the Jacobian of G is the matrix

$$\begin{bmatrix} \frac{\partial x}{\partial u}, & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u}, & \frac{\partial y}{\partial v} \end{bmatrix}.$$

We usually care about the absolute value of the determinant, $\mathrm{Jac}(G)$ ($\frac{\partial(x,y)}{\partial(u,v)}$). (Context will distinguish which we mean.)

Ex: $G(u,v) = (u^3+v, uv)$. Find the Jacobian, the absolute value of its determinant, and evaluate it $(2,1)$.

Sol: $\begin{bmatrix} \frac{\partial x}{\partial u}, & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u}, & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} 3u^2, & 1 \\ v, & u \end{bmatrix}$. The abs. val. of the det. is

$|3u^3 - v|$. At $(2,1)$, this is $3 \cdot 2^3 - 1 = 23$.

$\boxed{\text{Useful Fact:}}$ Consider a linear map $G(u,v) = (Au + Bv, Cu + Dv)$. Then its Jacobian is $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, and the determinant of this is $AD - BC$. The abs. val. is then $|AD - BC|$. It will then follow from what we do later that $\mathrm{Area}(G(D)) = |AD - BC| \, \mathrm{Area}(D)$ for any region D.

For a nonlinear map, the abs. val. of the det. of the Jacobian is in general a function of u and v. So this equation doesn't really make sense for non linear maps. But a version of it is true.

$$|Jac(G)(P)| = \lim_{|D|\to 0} \frac{Area(G(D))}{Area(D)}$$

For ~~some~~ any point P in D, where $|D| = max(\sqrt{x^2+y^2} : x, y \text{ in } D)$. This will be useful in the future.

(Essentially this means that for a "small" D, the Jacobian evaluated at a point in D can be used to approximate the area of G(D).

10/8 - Change of Variables 2

A map G is one-to-one, or injective, if whenever $G(u_0, v_0) = G(u_1, v_1)$, $u_0 = u_1$ and $v_0 = v_1$.

Ex: $(r_1 \cos\theta_1, r_1 \sin\theta_1) = (r_0 \cos\theta_0, r_0 \sin\theta_0)$.

Then $r_1^2 = (r_1 \cos\theta_1)^2 + (r_1 \sin\theta_1)^2 = (r_0 \cos\theta_0)^2 + (r_0 \sin\theta_0)^2 = r_0^2$, so $r_0 = r_1$. Then $\cos\theta_1 = \cos\theta_0$ and $\sin\theta_1 = \sin\theta_0$, so $\theta_0 = \theta_1$.

Change of Variables Formula:

If $G(u,v)$ is one-to-one on the interior of D, and it has continuous partial derivatives, then

$$\iint_D f(x(u,v), y(u,v)) \, |Jac(G)(u,v)| \, du\,dv =$$

$$\iint_{G(D)} f(x,y) \, dx\,dy.$$

Ex: $P(r,\theta) = (r\cos\theta, r\sin\theta)$.

$$|Jac(G)| = \left| \begin{bmatrix} \cos\theta, & -r\sin\theta \\ \sin\theta, & r\cos\theta \end{bmatrix} \right| = r\cos^2\theta + r\sin^2\theta = r$$

$$\iint_D f(x(r,\theta), y(r,\theta)) \cdot r \, dr\,d\theta = \iint_{G(P)} f(x,y) \, dx\,dy.$$

Ex: $G(u,v) = (u/v,\ uv)$, $D = [1,2] \times [1,2]$. Compute

$$\iint_{G(D)} (x^2+y^2)\,dx\,dy$$

Sol: $\displaystyle \iint_{G(D)} x^2+y^2\,dx\,dy = \int_1^2\int_1^2 x(u,v)^2 + y(u,v)^2\ |\mathrm{Jac}(G)(u,v)|\,du\,dv$

$$= \int_1^2\int_1^2 \frac{u^2}{v^2} - u^2v^2\ |\mathrm{Jac}(G)(u,v)|\,du\,dv$$

the Jacobian is given by

$$\begin{bmatrix} 1/v, & -u/v^2 \\ v, & u \end{bmatrix} = \frac{u}{v} + \frac{u}{v} = \frac{2u}{v}.$$ Therefore, we have

$$\int_1^2\int_1^2 \left(\frac{u^2}{v^2} - u^2v^2\right)\frac{2u}{v}\,du\,dv = \int_1^2\int_1^2 \frac{2u^3}{v^3} - 2u^3v\ d\,dv =$$

$$\int_1^2 \frac{u^4}{2v^3} - \frac{u^3v}{2}\Big|_1^2\,dv = \int_1^2 \frac{15}{2v^3} - \frac{7v}{2}\,dv = \frac{-15}{4v^2} + \frac{7v^2}{4}\Big|_1^2$$

$$7 - \frac{15}{16} + \frac{15}{4} - \frac{7}{4} = 7 - \frac{15}{16} + 8/4 = 7 - \frac{15}{16} + \frac{32}{4} = 7 + \frac{17}{4} =$$

$45/4$.

If $G: D \to R$, then $G^{-1}: R \to G$, the inverse of G, means that $F(x,y) = (u,v)$, where $G(u,v) = (x,y)$. Note that this only makes sense when G is one-to-one.

Fact: $\mathrm{Jac}(G^{-1}) = \dfrac{1}{\mathrm{Jac}(G)}$ whenever $\mathrm{Jac}(G)$ is not 0. (In general, if G can be inverted, then $\mathrm{Jac}(G) \neq 0$)