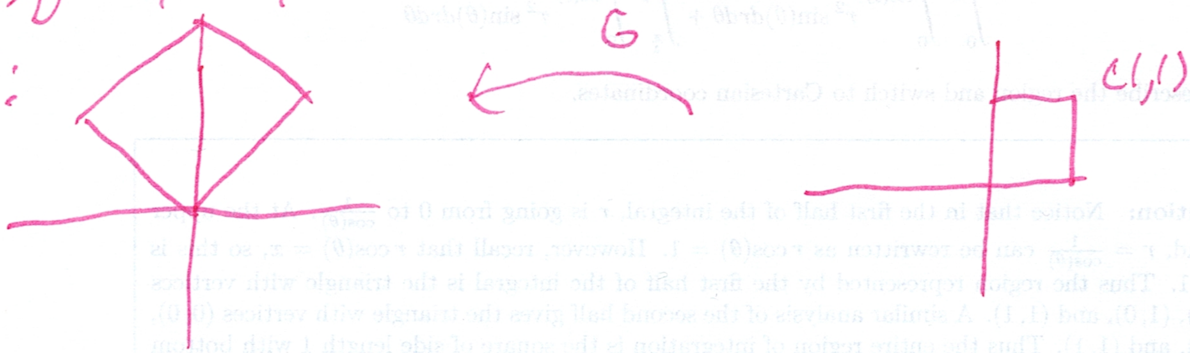
Ex: Let D be the square with vertices $(0,0)$, $(-1,1)$, $(1,1)$, and $(0,2)$. Calculate $\iint_D x^2-y^2\,dA$ using change of variables.

Sol:



Want $G(1,0)=(1,1)$: $x(u,v)=u+f(v)$, $y(u,v)=u+h(v)$.

Want $G(0,1)=(-1,1)$: $x(u,v)=u-v$, $y(u,v)=u+v$.

Therefore $G(u,v)=(u-v,\ u+v)$.

Notice that $x^2-y^2=(x+y)(x-y)$, so our function to integrate is $-4uv$.

The Jacobian is $\left|\det\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\right| = 2$.

Therefore $\iint_D x^2-y^2\,dA = \int_0^1\int_0^1 -8uv\,du\,dv = \int_0^1 -4u^2v\Big|_0^1\,dv =$

$\int_0^1 -4v\,dv = -2v^2\Big|_0^1 = -2$

If $G: D \to R$, then $G^{-1}: R \to D$, the inverse of G, means that $G^{-1}(x,y)=(u,v)$, where $G(u,v)=(x,y)$.
Note that this only makes sense when G is one-to-one.

Fact: $\mathrm{Jac}(G^{-1}) = \dfrac{1}{\mathrm{Jac}(G)}$ whenever $\mathrm{Jac}(G) \neq 0$.

Ex: Compute $\iint_D xy(x^2+y^2)\,dx\,dy$ over the region D defined via $-3 \le x^2 - y^2 \le 3$, and $1 \le xy \le 4$.

Sol: Let F be $u = x^2 - y^2$ and $v = xy$. Then $F(D) = [-3,3] \times [1,4]$ in the uv-plane.

$$\iint_{F(D)} f(x(u,v), v(u,v))\,|\mathrm{Jac}(G)|\,du\,dv = \iint_D f(x,y)\,dx\,dy$$

$$\parallel$$

$$\iint_{F(D)} f(x(u,v), v(u,v))\,\frac{1}{|\mathrm{Jac}(F)|}\,du\,dv$$

$$\begin{bmatrix} 2x & -2y \\ y & x \end{bmatrix} = 2x^2+2y^2.$$

$$\iint_{F(D)} xy\,\cancel{(x^2+y^2)}\,\frac{1}{2\cancel{(x^2+y^2)}}\,du\,dv = \iint_{F(D)} \frac{xy}{2}\,du\,dv$$

$$= \int_1^4\int_{-3}^{3} \frac{v}{2}\,du\,dv = \int_1^4 3v\,dv = \frac{3v^2}{2}\Big|_1^4 = 22\tfrac{1}{2}.$$

Ex: Consider the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and let D be its interior. Use change of variables to compute the area of D.

Sol: Let $u=\frac{x}{a}$ and $v=\frac{y}{b}$. Then

$$G(u,v)=(au,\ bv),\quad \text{and}\quad G^{-1}(D)=u^2+v^2=1.$$

Furthermore, $|Jac(G)|=\left|\det\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix}\right|=|ab|$,

Thus $$\iint_D 1\,dA=\iint_{G^{-1}(D)} ab\,dA=\int_0^{2\pi}\int_0^1 abr\,dr\,d\theta$$
$$=ab\int_0^{2\pi}\frac{r^2}{2}\Big|_0^1 d\theta=\frac{ab}{2}\int_0^{2\pi}1\,d\theta=ab\pi.$$

Change of variables in three dimensions

If $G: E\to\mathbb{R}^3$ is one-to-one on E, then

$x(u,v,w)\quad y(u,v,w)\quad z(u,v,w)$

$$\iiint_E f(x,y,z)\,|Jac(G)|\,dV=\iiint_{G(E)} f(x,y,z)\,dV,$$

where $$|Jac(G)|=\left|\det\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{pmatrix}\right|.$$

Ex: $C(r,\theta,\cancel{\phi}\,z)=(r\cos\theta,\ r\sin\theta,\ z)$.

$$|Jac(C)|=\left|\det\begin{pmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\right|=$$

$$|r\cos^2\theta-(-r\sin^2\theta)+0|=|r(\cos^2\theta+\sin^2\theta)|=r,$$