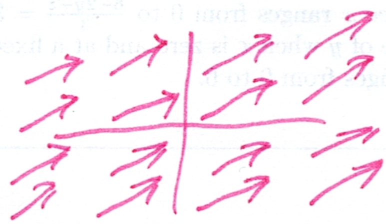
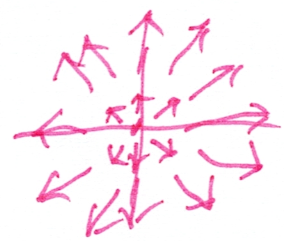
10/13 - Vector Fields

A vector field is a function that assigns vectors to points. These can be two dimensional or three-dimensional.

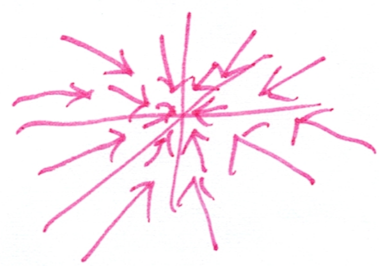
Ex: $F(x,y) = \langle 3, 2 \rangle$



Ex: $F(x,y) = x\hat{i} + y\hat{j}$



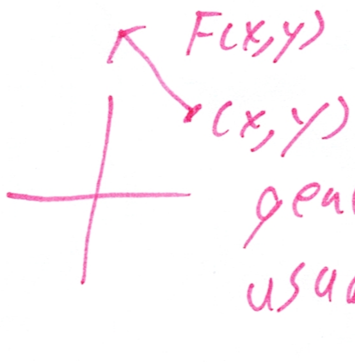
Ex: $F(x,y,z) = \langle -x, -y, -z \rangle$



A vector field is continuous if all of its component functions are continuous.

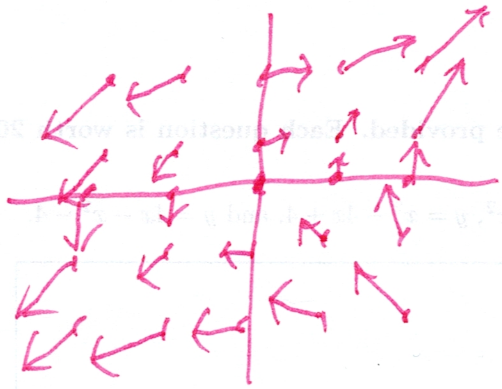
To draw a vector field, we draw the vector $F(x,y)$ with its starting point at (x,y).

Ex:



We plot as many as we need to understand the general shape. It is usually easiest to use a grid to keep things clean and readable.

Ex: $F(x,y) = \langle 2y, 2x \rangle$



A radial vector field is of the form

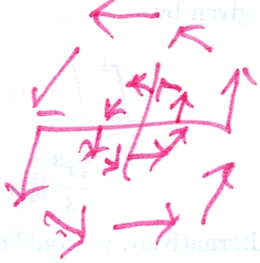
$F(x,y,z) = \langle cx, cy, cz \rangle$

for some $c \neq 0$.

(Also in 2D)

More generally, a radial vector fields points towards ~~Ex: A rotational vector field~~ or away from the origin. This can also be phrased as being perpendicular to a circle at the origin at every point.

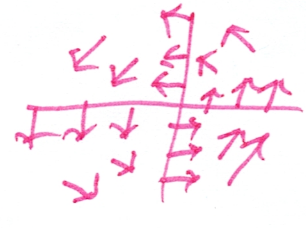
Conversely, a rotational vector field is one that is tangent (or parrellel) everywhere to the circle at the origin



Ex: $F(x,y) = \langle -y, x \rangle$.

A <u>unit</u> vector field is one where $\|F(x,y,z)\|$ (the length of $F(x,y,z)$) is 1 everywhere.

Ex: $F(x,y) = \left\langle \frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}} \right\rangle$



Scaling a vector to make its length 1 is called <u>normalizing</u>. Given any vector field $F(x,y,z)$, the vector field $\frac{F(x,y,z)}{\|F(x,y,z)\|}$ is a unit vector field, the normalization of F.

A <u>gradient</u> vector field is a vector ~~field~~ such that the components are the partial derivatives of some scalar function f.

Ex: $f(x,y,z) = x^3y^2z$

$$\nabla f = \langle 3x^2y^2z,\ 2x^3yz,\ x^3y^2 \rangle$$

Drawing a picture of a gradient vector field helps us determine properties of the original <u>(potential)</u> function.

Ex: Find the ~~maximum~~ minimum of $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}$

Sol: $\nabla f = \langle x, y, z \rangle$

$(0,0)$

Maxes occur where vectors come together.

Mins occur when they spread apart.

Saddle points occur when they come together in one direction but split in another

A ~~function~~ curve $r(t)$ is a flow line of a vector field F if the vector field is its derivative at every point, i.e. $r'(t) = F(r(t))$.

Ex: The unit circle is a flow line of $F(x,y) = \langle -y, x \rangle$.

Sol: The unit circle is given by $\langle \cos(t), \sin(t) \rangle$.

$r'(t) = \langle -\sin(t), \cos(t) \rangle$. $F(r(t)) = \langle -\sin(t), \cos(t) \rangle$.