10/15 - Conservative Vector Fields

Recall that a vector field is conservative if it is a gradient vector field, i.e. $F = \nabla f$ for some continuous, differentiable function f. f is called a potential function, and it is unique up to a constant.

We can find potential functions using integration.

Ex: Find the potential function f for $F(x,y,z) = \langle 2x\cos(yz), -x^2 z\sin(yz) + 2yz^2, -x^2 y\sin(yz) + 2zy^2 \rangle$ and $f(0,0,0) = 0$.

Sol: $f_x = 2x\cos(yz)$: $\int 2x\cos(yz)\,dx = x^2\cos(yz) + g(y,z)$

$f_y = -x^2 z\sin(yz) + 2yz^2$ AND $f_y = -x^2 z\sin(yz) + \frac{\partial g}{\partial y}(y,z)$

Therefore, $\frac{\partial g}{\partial y}(y,z) = 2yz^2$.

Thus we have $g(y,z) = \int 2yz^2\,dy = y^2z^2 + h(z)$.

Finally, $f_z = -x^2 y\sin(yz) + 2zy^2$ AND $-x^2 y\sin(yz) + 2zy^2 + h'(z)$.

Then $h'(z) = 0$, so $h(z) = \int 0\,dz = C$.

So $f = x^2\cos(yz) + 2yz^2 + C$. Given $f(0,0,0)=1$, we have $0 + 0 + C = 0$, so $C = 0$.

One way to see a vector field is <u>not</u> conservative is for this process to fail.

Ex: $F(x,y) = \langle y, -x \rangle$. If it had a potential function f, then $f_x = y$. Therefore, $f = \int y\,dx = xy + g(y)$. But then $f_y = -x$ and $f_y = x + g'(y)$. This implies that $g'(y) = -2x$, which is impossible because $g(y)$ does not depend on x.

If we boil this process down into what needs to happen for the integration process to succeed, we get the following test for conservativity:

A vector field $F(x,y) = \langle P(x,y), Q(x,y) \rangle$ is conservative if and only if $P_y = Q_x$.

A vector field $F(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$ is conservative if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$, and $\frac{\partial R}{\partial x} = \frac{\partial P}{\partial x}$.

Ex: $F(x,y) = \langle y, -x \rangle$. Then $P_y = 1$, but $Q_x = -1$.

Recall that ∇ is the "vector" $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$ in 3D and $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle$ in 2D.

The <u>divergence</u> of F in 2D or 3D is "$\nabla \cdot F$": $\operatorname{div} F = P_x + Q_y$ (2D)

$\operatorname{div} F = P_x + Q_y + R_z$ (3D)

It represents inflow and outflow! negative means there's a net gain at a point, positive means a net loss.

Ex: $F(x,y,z) = \langle -x, -y, -z \rangle$.

$\operatorname{div} F = \nabla \cdot F = -1 + -1 + -1 = -3.$

A <u>sink</u> is a point with negative divergence, a <u>source</u> has positive div.

The <u>curl</u> of F is only defined in 3D technically, but we can modify it for 2D. It is "$\nabla \times F$":

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (R_y - Q_z)i - (R_x - P_z)j + (Q_x - P_y)k,$$

For a 2D vector field, we do the curl of the vector field extended to 3D by a zero z-component:

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \cancel{(0-0)i - (0-0)j +} \underline{(Q_x - P_y)k}.$$

Curl measures how much something rotates about a point: Think drains.