The divergence of $F$ is:

$F(x,y) = \langle P(x,y), Q(x,y) \rangle$ : $\text{div} F = P_x + Q_y$

$F(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$ : $\text{div} F = P_x + Q_y + R_z$

This represents how much is flowing out of a given point. Negative divergence means flowing in.

A <u>sink</u> has negative divergence

A <u>source</u> has positive divergence

Ex: $F(x,y,z) = \langle x^2, y, -z \rangle$

$\text{div} F = 2x + 1 - 1 = 2x$

Therefore the sinks are $(x,y,z)$ with $x<0$, and the sources are $(x,y,z)$ with $x>0$.

"$\underline{\text{div} F = \nabla \cdot F}$"

The curl of $F$ is "$\nabla \times F$."

If $F$ is 2D, we add a zero as the z-component.

A vector field $F$ is conservative if and only if its curl is zero everywhere.

10/18 - Scalar Line Integrals

For line integrals, we are given a parametrized curve $C$ of the form $r(t) = \langle x(t), y(t), z(t) \rangle$, $a \le t \le b$. Then

$$\int_C f \cdot ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$$

In other words,

$$\int_C f \cdot ds = \int_a^b f(x(t), y(t), z(t)) \cdot \|r'(t)\|\, dt.$$

The $\|r'(t)\|$ comes from the length of a tiny piece of $C$ at $t$. It is like Jacobians, scaling up $dt$ to match the size of a piece of $C$.

Ex: $r(t) = \langle t^2, 3t^2 \rangle$ $0 \le t \le 4$.

Calculate $\int_C x + y\, ds$.

Sol: $\int_C x + y\, ds = \int_0^4 t^2 + 3t^2\, ds$.

$ds = \|r'(t)\|\, dt = \sqrt{(2t)^2 + (6t)^2}\, dt = \sqrt{4t^2 + 36t^2}\, dt = 2\sqrt{10}\, t\, dt$

$\int_0^4 8\sqrt{10}\, t^3\, dt = 2\sqrt{10}\, t^4 \Big|_0^4 = 512\sqrt{10}.$

Ex: Let $C$ be the path which traverses the unit circle 1.5 times at regular speed. Calculate $\int_C x^2\,ds$.

Sol: $r(t) = \langle \cos(t), \sin(t) \rangle$, $0 \le t \le 3\pi$.

$$\int_C x^2\,ds = \int_0^{3\pi} \cos^2 t\, \|r'(t)\|\,dt = \int_0^{3\pi} \cos^2 t \sqrt{(-\sin t)^2 + \cos^2 t}\,dt =$$

$$\int_0^{3\pi} \cos^2 t\,dt = \frac{t}{2} + \frac{\sin 2t}{4}\Big|_0^{3\pi} = \frac{3\pi}{2}.$$

Scalar line integrals can be used to compute the lengths of curves.

Ex. Compute the length of the helix $r(t) = \langle 2\cos(2t), 2\sin(2t), 3t \rangle$, $0 \le t \le 4\pi$.

Sol: The length is given by $\int_C 1\,ds$.

$$\int_C 1\,ds = \int_0^{4\pi} \|r'(t)\|\,dt = \int_0^{4\pi} \sqrt{(-4\sin 2t)^2 + (4\cos 2t)^2 + 9}\,dt$$

$$= \int_0^{4\pi} 5\,dt = 20\pi.$$