10/20 - Vector Line Integrals

Given a vector field F and a path $C = r(t)$, we think of F as a force acting on a particle moving along C. Then the work done by the field in moving the particle is

$$\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\,dt$$

Ex: Calculate the work done by the vector field $F(x,y,z) = \langle 3, 2, z \rangle$ on the particle given by $r(t) = (t, t^2, t+2)$ $0 \le t \le 5$

Sol: $r'(t) = \langle 1, 2t, 1 \rangle$, so we have

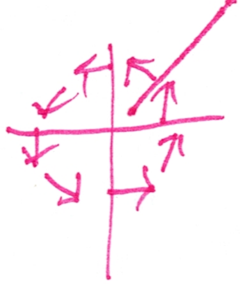
$$\int_C F \cdot dr = \int_0^5 \langle 3, 2, z \rangle \cdot \langle 1, 2t, 1 \rangle\,dt = \int_0^5 \langle 3, 2, t+2 \rangle \cdot \langle 1, 2t, 1 \rangle\,dt$$

$$= \int_0^5 3 + 4t + t + 2\,dt = \int_0^5 5t + 5\,dt = \frac{5t^2}{2} + 5t \Big|_0^5 = \frac{125}{2} + 25$$

It is important to note that the vector line integral through a vector field is <u>NOT</u> the total work done! It is only the work done by the vector field.

Ex: $F(x,y) = \langle -y, x \rangle$, $r(t) = \langle t, t \rangle$, $a \le t \le b$. Then

$$\int_C F \cdot dr = \int_a^b \langle -y, x \rangle \cdot \langle t, t \rangle\,dt = \int_a^b \langle -t, t \rangle \cdot \langle t, t \rangle\,dt = 0.$$



The particle is always moving perpendicular to the vector field, so the ~~net~~ work done by the vector field is 0.

Given a curve C connecting ~~going from~~ (a,b) to (c,d), there are two ways to traverse it. We can go from (a,b) to (c,d), or from (c,d) to (a,b). This is the orientation of C.

If C has one orientation, −C represents the same curve with the opposite orientation, and $\int_{-C} F\cdot dr = -\int_C F\cdot dr$.

Ex: $F(x,y,z) = \langle 3, 2, z\rangle$, $r(t) = (3-t, (5-t)^2, 7-t)$, $0 \le t \le 5$

Then $\int_C F\cdot ds = \int_0^5 \langle 3, 2, 7-t\rangle \cdot \langle -1, 2t-10, -1\rangle \, dt$

$= \int_0^5 -3 + 4t - 20 + t - 7 \, dt = \int_0^5 5t - 30 \, dt$

$= \frac{5t^2}{2} - 30t \Big|_0^5 = \frac{125}{2} - 150 = \frac{125}{2} - \frac{300}{2} =$

$-\frac{175}{2} = -\left(\frac{125}{2} + 25\right).$

Sometimes these are written like

$$\int_C P\,dx + Q\,dy + R\,dz$$

We integrate these by switching to t,

Ex: $\int_C x^2 dx + y\,dy - z^3 dz$, $r(t) = \langle t, t^2, t^3 \rangle$ $0 \le t \le 1$

Sol: $x^2 dx = t^2 dt$, $y\,dy = \cancel{t}^2 \cdot 2\cancel{t}\,dt = 2t^3 dt$

$z\,dz = t^3 \cdot 3t^2 dt = 3t^5 dt$

$$\int_C x^2 dx + y\,dy - z^3 dz = \int_0^1 t^2 + 2t^3 dt \quad 3t^{\cancel{5}} dt =$$

$$\frac{t^3}{3} + \frac{t^4}{2} + \frac{t^{\cancel{6}2}}{\cancel{4}}\Big|_0^1 = \frac{1}{3} + \frac{1}{2} + \frac{1}{\cancel{4}} = \frac{13}{12}$$

The Flux integral across a curve in 2D C is $\int_C F \cdot N\,ds$, which is $\int_C F \cdot \langle r_y(t), -r_x(t) \rangle dt$,

Ex: $F = \langle 2x, 2y \rangle$, C is given by $r(t) = \langle \cos(t), \sin(t) \rangle$, $0 \le t \le 2\pi$

$\frac{d}{dt}\sin(t) \quad -\frac{d}{dt}\cos(t)$

$$\int_C F \cdot N\,ds = \int_0^{2\pi} \langle 2x, 2y \rangle \cdot \langle \cos(t), \sin(t) \rangle\, dt =$$

$$\int_0^{2\pi} 2\cos^2 t + 2\sin^2 t\, dt = \int_0^{2\pi} 2\,dt = 4\pi$$