10/22 – Vector Fields and Line Integrals

Fundamental Theorem for Line Integrals:

If C is a smooth curve with parametrization $r(t)$, $a \leq t \leq b$, then if if f is differentiable,

$$\int_C \nabla f \cdot dr = f(r(b)) - f(r(a)),$$

In particular, if F is a conservative vector field, it has a potential function f. Then

$$\int_C F \cdot dr = f(r(b)) - f(r(a)).$$

Ex: $F(x,y,z) = \langle 2x\ln y, \frac{x^2}{y} + z^2, 2yz \rangle$, C given by $r(t) = \langle t^2, t, t \rangle$, $1 \leq t \leq e$.

Sol: F is conservative, so we find its potential function:

$f_x = 2x\ln y$, so $f = \int 2x\ln y \, dx = x^2 \ln y + g(y,z)$

$f_y = \frac{x^2}{y} + z^2$ and $f_y = \frac{x^2}{y} + \frac{\partial}{\partial y} g(y,z)$, so

$$\frac{\partial}{\partial y} g(y,z) = z^2.$$

Therefore $g(y,z) = \int z^2 \, dy = yz^2 + h(z)$.

$f_z = 2yz$ and $f_z = 2yz + h'(z)$,

so $h'(z) = 0$ and $f = x^2 \ln y + yz^2 + C$.

Therefore, $\int_C F\cdot dr = (x^2 \ln y + yz^2)\Big|_{(1,1,1)}^{(e^2,e,e)} =$

$$e^4 + e^3 - 1.$$

Notice that if we tried doing the integration, we'd get

$$\int_C F\cdot dr = \int_1^e \langle 2x\ln y, \frac{x^2}{y}+z^2, 2yz\rangle \cdot \langle 2t,1,1\rangle dt$$
$$= \int_1^e 4t^3 \ln t + t^3 + t^2 + 2t^2\, dt$$
$$= \int_1^e 4t^3 \ln t + t^3 + 3t^2\, dt$$

this is possible, but it is messy and requires integration by parts.

C is closed if it starts and ends at the same point. Then $\oint_C F\cdot dr = \int_C F\cdot dr$ is used to convey this property. $\oint_C F\cdot dr$ is called the circulation of F along C.

If F is conservative, then the circulation along any smooth closed curve is 0: $\oint_C F\cdot dr = \int_C F\cdot dr = F(r(b)) - F(r(a))$
$= F(r(b)) - F(r(b))$
$= 0.$

Recall that F is independent of path if $\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr$ whenever C_1 and C_2 have the same endpoints.

Hence, F is conservative if and only if it is independent of path.

So far we have only really thought about vector fields with domain $\mathbb{R}^3$. However, we can think about others.

<u>Note:</u> $\operatorname{curl} F = \vec{0}$ is equivalent to conservativity only when the domain of F is open and has no holes in it.

Ex: $F(x,y) = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$ in $\mathbb{R}^2$ with the origin removed. Then

$$\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2}$$

$$= \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2)(1) - (x)(2x)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

Thus $\operatorname{curl} F = \frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2} = 0$, but

$\int_C F \cdot dr$, where $C = r(t) = (\cos t, \sin(t))$, $0 \le t \le 2\pi$,

$= \int_0^{2\pi} \langle -\sin(t), \cos(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle \, dt =$

$\int_0^{2\pi} \sin^2 t + \cos^2 t \, dt = \int_0^{2\pi} 1 \, dt = 2\pi \neq 0.$