10/27 - Parametrized surfaces

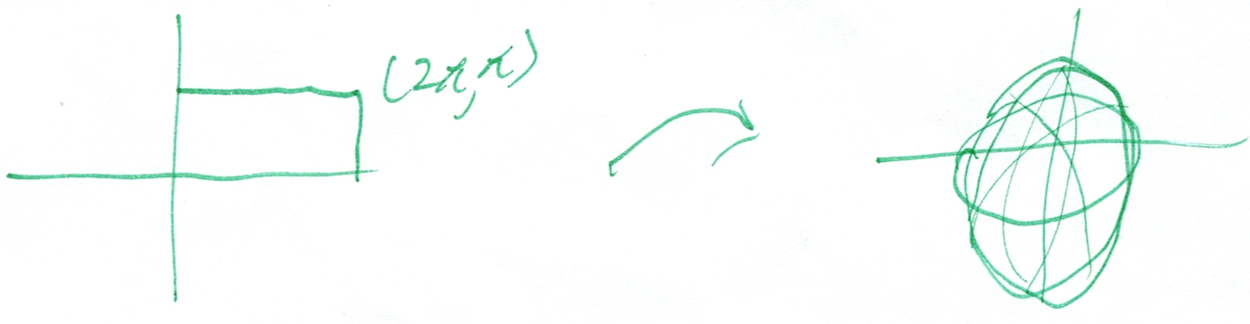
A parametrized surface S is of the form $\vec{r}(s,t) = \langle x(s,t), y(s,t), z(s,t) \rangle$

for $(s,t) \in D$

In other words, we are describing points in 3d using two variables like parametrized curves describe points in 2/3D using 1 variable.

D is the parameter domain, a shape in the s,t-plane.

Ex: $S(\theta,\phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$



$S(\theta,\phi)$ is the unit sphere.

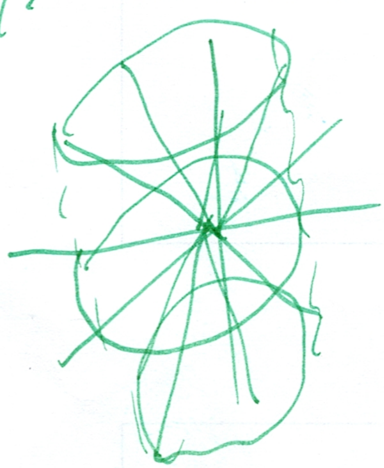
If we want the sphere of radius R, we use $(R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi)$

Given $z=f(x,y)$, we can always parametrize the graph of this function as $(x, y, f(x,y))$ (or $(s,t,f(s,t))$).

Ex: The parametrization of the paraboloid $f(x,y)=x^2+y^2$ is (x,y,x^2+y^2).

Ex: Give a parametrization of the cone $z^2=x^2+y^2$ which is inside the cylinder $x^2+y^2\leq 4$.

Sol:



Notice that a point on the upper half of the cone is $z=\sqrt{x^2+y^2}$, or $z=r$.

On the lower half,

$$z=-\sqrt{x^2+y^2}, \text{ or } z=-r.$$

Thus we can do

$$C(s,t)=(s\cos t, s\sin t, s),$$

$$-2\leq s\leq 2,\ 0\leq t\leq 2\pi.$$

In cylindrical coordinates, r cannot be negative. When parametrizing a surface, though, we can use "negative r" values if we want.

8. Compute $\int\int_D y^2 - x^2 dA$ using the change of variables $u = y - x$ and $v = x + y$, where D is the parallelogram with vertices $(0,0)$, $(1,1)$, $(-1,1)$, and $(0,2)$.

Solution: We first need to find the image of $\mathcal{D}$ under this change of variables. As the map is linear, it suffices to check the image of $(1,1)$ and $(-1,1)$. For the former, $u = 0$ and $v = 2$. For the latter, $u = 2$ and $v = 0$. Therefore, the image of $\mathcal{D}$ is $[0,2] \times [0,2]$.

Next we need to find the Jacobian of the inverse map, so we first compute the Jacobian of the given map. It is -2, and the absolute value is 2, so the absolute value of the Jacobian of the inverse map is $\frac{1}{2}$.

Therefore, as $y^2 - x^2 = (y-x)(y+x) = uv$,

$$\int\int_D x^2 - y^2 dA = \int_0^2 \int_0^2 uv\frac{1}{2}dudv = \int_0^2 \frac{u^2 v}{4}|_0^2 dv = \int_0^2 vdv = \frac{v^2}{2}|_0^2 = 2$$

$$N = T_s \times T_t$$

This is called the normal vector to the surface (at a point).

If $N(s_0, t_0) \neq \vec{0}$ for any (s_0, t_0) in the parametrized domain, we call our parametrization <u>regular</u>.

Ex: $S(\theta, \phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$

$$S_\theta = \langle -\sin\theta\sin\phi, \cos\theta\sin\phi, 0 \rangle$$

$$S_\phi = \langle \cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi \rangle$$

$$N = S_\theta \times S_\phi = \begin{pmatrix} i & j & k \\ -\sin\theta\sin\phi, & \cos\theta\sin\phi, & 0 \\ \cos\theta\cos\phi, & \sin\theta\cos\phi, & -\sin\phi \end{pmatrix} =$$

$$-\cos\theta\sin^2\phi\, i - (\sin\theta\sin^2\phi)\, j + (-\sin^2\theta\sin\phi\cos\phi - \cos^2\theta\sin\phi\cos\phi)\, k$$

$$= \langle -\cos\theta\sin^2\phi, -\sin\theta\sin^2\phi, -\sin\phi\cos\phi \rangle$$

$$= -\sin\phi \langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi \rangle$$

$$= -\sin\phi \langle x, y, z \rangle$$

<u>Not</u> regular! $\sin(0) = \sin(\pi) = 0$.

(Note $S_\phi \times S_\theta$ flips the sign! inwards vs outwards orientation!)