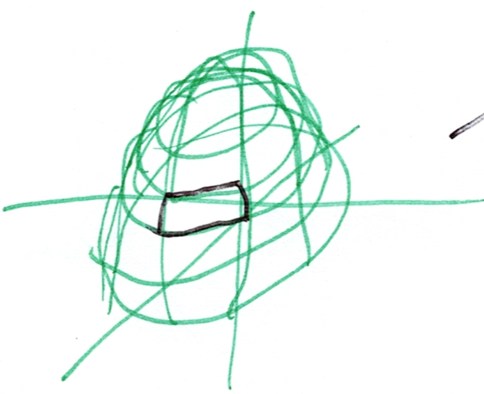
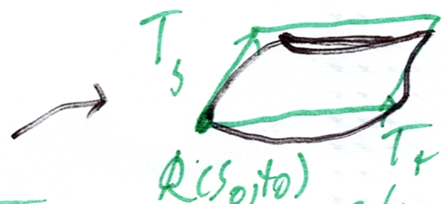
10/29 - Surface Integrals

We want to integrate over a surface S. The grid curves cut it into small chunks.





At the point $R(s_0,t_0)$, we have a shape which is a bent parallelogram, with one side given by T_s and the other by T_t. In the limit, this will have the same area as the parallelogram, which is $\|T_s \times T_t\| = \|N\|$.

Integral over a parametrized surface:

$$\iint_S f(x,y,z)\,dA = \iint_D f(R(s,t))\,\|N\|\,ds\,dt$$

That means surface area is given by $\iint_D \|N\|\,ds\,dt$.

Ex: Calculate the surface area of the part of the cone $z^2 = x^2+y^2$ above $9 = x^2+y^2$

Sol: $R(\theta, z) = (z\cos\theta, z\sin\theta, z)$ $\quad 0 \le z \le 3, \quad 0 \le \theta \le 2\pi$.

$R_\theta = \langle -z\sin\theta, z\cos\theta, 0 \rangle$

$R_z = \langle \cos\theta, \sin\theta, 1 \rangle$

$N = R_\theta \times R_z = \langle z\cos\theta, z\sin\theta, -z \rangle$

$\|N\| = \sqrt{z^2\cos^2\theta + z^2\sin^2\theta + z^2} = z\sqrt{2}$.

Surface Area is: $\iint_D \|N\| \, d\theta \, dt =$

$$\int_0^{2\pi}\int_0^{3} z\sqrt{2}\, dz\, d\theta = \int_0^{2\pi} \frac{\sqrt{2} z^2}{2}\Big|_0^3 d\theta =$$

$$\int_0^{2\pi} \frac{9}{2}\sqrt{2}\, d\theta = 9\sqrt{2}\pi.$$

Ex: Surface integrals can be used to compute quantities like mass or charge.

Ex: Suppose a sphere of radius 2 has charge $3z^2$ at every point on its surface. Find the total charge.

Sol: $\iint_S 3x^2 \, dA = \iint_D 3\cos^2\phi \, \|N\| \, d\theta \, d\phi$

$\|N\| = 4\sin\phi$

$$\int_0^{2\pi}\int_0^{\pi} 48\cos^2\phi \sin\phi \, d\phi \, d\theta = \int_0^{2\pi} -16\cos^3\phi \Big|_0^{\pi} d\theta =$$

64π.

If our parametrized surface is of the form $R(x,y) = (x, y, f(x,y))$, then

$$R_x = \langle 1, 0, \tfrac{\partial}{\partial x} f(x,y) \rangle$$

$$R_y = \langle 0, 1, \tfrac{\partial}{\partial y} f(x,y) \rangle$$

$$N = \langle -\tfrac{\partial}{\partial x} f(x,y), -\tfrac{\partial}{\partial y} f(x,y), 1 \rangle$$

Then $\|N\| = \sqrt{\left(\tfrac{\partial}{\partial x} f(x,y)\right)^2 + \left(\tfrac{\partial}{\partial y} f(x,y)\right)^2 + 1} = \sqrt{f_x^2 + f_y^2 + 1}$

Ex: Compute $\iint_S 4\, dA$, where S is the part of the paraboloid $z = 1 - x^2 - y^2$ with $z \geq 0$.

Sol: $\|N\| = \sqrt{1 + 4x^2 + 4y^2}$

$\iint_S 4\, dA = \iint_D 4\sqrt{1 + 4x^2 + 4y^2}\, dA$, where D is the unit disk $x^2 + y^2 \leq 1$. Switching to polar, we get

$$\int_0^{2\pi}\int_0^1 4r\sqrt{1+4r^2}\, dr\, d\theta = \int_0^{2\pi} \frac{\sqrt{1+4r^2}}{2}\Big|_0^1 d\theta =$$

$$\int_0^{2\pi} \frac{\sqrt{5}-1}{2}\, d\theta = (\sqrt{5}-1)\pi.$$