11/01 – Vector Field Surface Integrals

Recall that Given a parametrized surface $R(s,t)$, there are two choices of Normal vector: $T_s \times T_t$ or $T_t \times T_s$.

We say R is oriented positively if we take the outwards/upwards choice of normal vector.

(If N is continuous, we say R is orientable)

If $R(x,y) = \langle x, y, f(x,y) \rangle$, then the positive orientation is $N = \langle -f_x, -f_y, 1 \rangle$.

Not all surfaces are orientable.

Given a vector field F and a surface $R(s,t)$, the surface integral of F over R is

$$\iint_S F \cdot d\vec{S} = \iint_S F \cdot \hat{N}\, dS = \iint_D F \cdot N \, ds\, dt$$

This is called the flux of F through or across S, and is referred to as a flux integral.

Ex: Calculate $\iint_R F \cdot d\vec{S}$, where $F = \langle -y, x, 0 \rangle$ and

$R(s,t) = \langle s, t^2 - s, s + t \rangle$, $0 \le s \le 3$, $0 \le t \le 4$

Sol: $R_s = \langle 1, -1, 1 \rangle$

$R_T = \langle 0, 2t, 1 \rangle$

$R_s \times R_T = \langle -2t-1, -1, 2t \rangle$

This is the upwards Normal, so we use it,

$$\iint_R F \cdot d\vec{S} = \iint_D \langle -y, x, 0 \rangle \cdot \langle -2t-1, -1, 2t \rangle \, ds\, dt =$$

$$\int_0^4 \int_0^3 \langle s - t^2, s, 0 \rangle \cdot \langle -2t-1, -1, 2t \rangle \, ds\, dt =$$

$$\int_0^4 \int_0^3 -2st - s + 2t^3 + t^2 - s \, ds\, dt = \int_0^4 \int_0^3 2t^3 + t^2 - 2s - 2st \, ds\, dt$$

$$= \int_0^4 6t^3 + 3t^2 - (s^2 + s^2 t)\Big|_0^3 \, dt = \int_0^4 6t^3 + 3t^2 - 9t - 9 \, dt$$

$$= \left(\frac{3t^4}{2} + t^3 - \frac{9t^2}{2} - 9t \right)\Big|_0^4 = 384 + 64 - 72 - 36 = 340$$

Ex: Calculate the flux of $F = \langle 0, -z, y \rangle$ throught the portion of the unit sphere in the first octant, oriented outwards.

Sol: Recall that the normal vector for a sphere of radius ρ pointing outwards is given by $\rho \sin\phi \langle x, y, z \rangle$. Thus

$$\iint_S F \cdot d\vec{S} = \int_0^{\pi/2} \int_0^{\pi/2} \langle 0, -z, y \rangle \cdot \sin\phi \langle x, y, z \rangle \, d\phi\, d\theta =$$

$$\int_0^{\pi/2} \int_0^{\pi/2} (-zy + yz) \sin\phi \, d\phi\, d\theta = 0.$$

Flux integrals can be used to compute physical quantities like how much water flows through a surface.

Ex: Suppose water is flowing according to $\langle 2x, 2y, z\rangle$. How much water flows through the upper half of the sphere of radius 3?

Sol: $\iint_S \langle 2x, 2y, z\rangle \cdot 3\sin\phi \langle x, y, z\rangle \, d\vec{S} = \int_0^{2\pi}\int_0^{\pi/2} 3\sin\phi\,(2x^2+2y^2+z^2)\,d\phi\,d\theta$

$= \int_0^{2\pi}\int_0^{\pi/2} 3\sin\phi\,(2\sin^2\phi\cos^2\theta + 2\sin^2\phi\sin^2\theta + \cos^2\phi)\,d\phi\,d\theta =$

$= \int_0^{2\pi}\int_0^{\pi/2} 3\sin\phi\,(2\sin^2\phi + \cos^2\phi)\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/2} 3\sin\phi\,(\sin^2\phi + 1)\,d\phi\,d\theta$

$= \int_0^{2\pi}\int_0^{\pi/2} 3\sin^3\phi + 3\sin\phi\,d\phi\,d\theta = \int_0^{2\pi} 3\left(-\cos\phi + \frac{\cos^3\phi}{3}\right) - 3\cos\phi\Big|_0^{\pi/2} d\theta =$

$\int_0^{2\pi} 3\left(1 - \tfrac{1}{3}\right) + 3\,d\theta = \int_0^{2\pi} 3 - 1 + 3\,d\theta = \int_0^{2\pi} 5\,d\theta = 10\pi$