11/03 - Green's Theorem

Let C be a ccw-simple closed curve, and let D be its interior. Let $F = \langle P, Q \rangle$ be a vector field. Then:

Circulation Green's Theorem:

$$\oint_C F \cdot dr = \oint_C P\,dx + Q\,dy = \iint_D \text{"curl}(F)\text{"}\,dA = \iint_D (Q_x - P_y)\,dA$$

If C is clockwise, then we add a minus sign. The idea here is that $\oint_C F \cdot dr$ measures how much extra work is being done by a nonconservative vector field. curl F is a measure of how nonconservative a vector field is, so $\iint_D \text{curl}(F)\,dA$ is adding this up for all points in our region.

Flux Green's Theorem

$$\oint F \cdot N\,ds = \oint_C Q\,dx - P\,dy = \iint_D \text{div}(F)\,dA = \iint_D P_x + Q_y\,dA$$

The idea here is that $\oint_C F \cdot N\,ds$ is measuring how much stuff comes in or out of the shape, which div F measures at a specific point; $\iint_D \text{div}(F)\,dA$ adds up all these values.

Ex: Calculate $\int F \cdot dr$, where $F = \langle x^2y, y-3 \rangle$ and C is the rectangle path traversing the points (1,1), (4,1), (4,5), and (1,5) in that order.

Sol: By Green's Theorem, $\oint_C F \cdot dr = \iint_D Q_x - P_y \, dA$

$Q_x = 0$, $P_y = x^2$, $D = 1 \le x \le 4,\ 1 \le y \le 5$. therefore

$$\oint_C F \cdot dr = \int_1^5 \int_1^4 -x^2 \, dx\,dy = \int_1^5 -\frac{x^3}{3}\Big|_1^4 dy = \int_1^5 -21\,dy = -84.$$

We can double check this by calculating directly. C is split into 4 pieces:

$$C_1(t) = (t, 1), \quad 1 \le t \le 4$$
$$C_2(t) = (4, t), \quad 1 \le t \le 5$$
$$C_3(t) = (4-t, 5) \quad 0 \le t \le 3$$
$$C_4(t) = (1, 5-t) \quad 0 \le t \le 4$$

$$\int_{C_1} F \cdot dr = \int_{C_1} x^2y\,dx + (y-3)\,dy = \int_1^4 x^2\,dx = \frac{x^3}{3}\Big|_1^4 = 21$$

$$\int_{C_2} F \cdot dr = \int_{C_2} x^2y\,dx + (y-3)\,dy = \int_1^5 y-3\,dy = \frac{y^2}{2} - 3y\Big|_1^5 = \frac{25}{2} - 15 - \frac{1}{2} + 3 = 0$$

$$\int_{C_3} F \cdot dr = \int_{C_3} x^2y\,dx + (y-3)\,dy = \int_0^3 (4-t)^2 5\,dt = -\frac{(4-t)^3 \cdot 5}{3}\Big|_0^3 = -\frac{5}{3} + \frac{64 \cdot 5}{3} = -105$$

$$\int_{C_4} F \cdot dr = \int_{C_4} x^2y\,dx + (y-3)\,dy = \int_0^4 2-t\,dt = \left(2t - \frac{t^2}{2}\right)\Big|_0^4 = 8 - 8 = 0$$

$21 + (-105) = -84$

Planimeters work off of Green's Theorem: They use line integrals to calculate area. The area of D is $\iint_D 1\,dA$. If we pick $F = \langle P, Q \rangle$ such that $Q_x - P_y = 1$, then

$$\iint_D 1\,dA = \iint_D Q_x - P_y\,dA = \oint_C F \cdot dr$$

by Green's Theorem. For example, take $F = \langle 0, x \rangle$. Then $\oint_C F \cdot dr = \oint_C x\,dy$ gives the area of D.

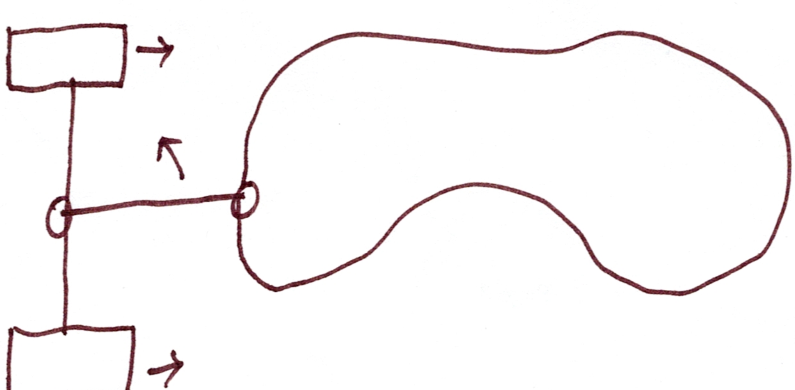
Ex: Calculate the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$.

Sol: The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can be parametrized by $r(t) = (a\cos t, b\sin t)$, $0 \leq t \leq 2\pi$. Then the area is given by $\oint_C F \cdot dr =$

$$\int_0^{2\pi} \langle 0, a\cos t \rangle \cdot \langle -a\sin t, b\cos t \rangle\,dt = \int_0^{2\pi} ab\cos^2 t\,dt =$$

$$ab\int_0^{2\pi} \cos^2 t\,dt = ab\left(\frac{t}{2} + \frac{\sin 2t}{4}\right)\Big|_0^{2\pi} = ab\pi.$$

There is nothing special about $\langle 0, x \rangle$. We can pick any vector field such that $Q_x - P_y = 1$.



(Planimeter: The wheels roll to measure the change in x. The change in angle is not used in the calculation.)