11/3 - Green's Theorem 2

Ex: Find the work done by $F(x,y) = \langle y+\sin x, e^y - x\rangle$ on a particle traversing the circle $x^2+y^2=4$ once in the clockwise direction, starting at $(2,0)$.

Sol: If we did this directly, we would get:

$$r(t) = (2\cos(-t), 2\sin(-t)) \quad -2\pi \le t \le 0$$

$$r'(t) = (2\sin t, -2\cos t)$$

$$F(r(t)) = \langle 2\sin(-t) + \sin(2\cos(-t)), e^{2\sin(-t)} - 2\cos(-t)\rangle$$

$$\text{Work} = \int_{-2\pi}^{0} \langle 2\sin\bar{t} + \sin(2\cos\bar{t}), e^{2\sin\bar{t}} - 2\cos\bar{t}\rangle \cdot \langle 2\sin t, -2\cos t\rangle\, dt$$

$$= \int_{-2\pi}^{0} 4\sin^2 t + 2\sin t\sin(2\cos\bar{t}) + 2\cos\bar{t}\, e^{2\sin\bar{t}} - 4\cos^2 t\, dt$$

$$= \int_{-2\pi}^{0} 4 + 2\sin t\sin(2\cos\bar{t}) + 2\cos t\, e^{2\sin\bar{t}}\, dt$$

$$= (4t + \cos(2\cos\bar{t}) + e^{2\sin\bar{t}})\Big|_{-2\pi}^{0} =$$

$$-\cos 2 + 1 - (-8\pi - \cos(2) + 1) = 8\pi$$

Conversely, by Green's theorem,

$$\int_C F\cdot dr = -\iint_D -2\, dA = \iint_D 2\, dA,$$ where

D is the disk $x^2+y^2\le 4$. This is

$$2\iint_D 1\, dA = 2\,\text{Area}(D) = 8\pi.$$

So far, all of our examples have been the circulation form of Green's theorem. Let's look at the flux form.

Ex: Compute the flux of $F(x,y) = \langle x, y\rangle$ across the boundary of square $[-1,1]\times[-1,1]$ oriented outwards.

Sol: $\text{Flux} = \int_C F\cdot\hat{N}\,ds = \int_C F\cdot N\,dt$. Like before, we split up the curve.

$C_1(t) = (-1, t)$ $\quad -1\le t\le 1 \quad r'(t) = \langle 0,1\rangle,\ N = \langle -1, 0\rangle$

$C_2(t) = (t, 1)$ $\quad -1\le t\le 1 \quad r'(t) = \langle 1,0\rangle,\ N = \langle 0, 1\rangle$

$C_3(t) = (1, 1-t)$ $\quad 0\le t\le 2 \quad r'(t) = \langle 0,-1\rangle,\ N = \langle 1, 0\rangle$

$C_4(t) = (1-t, -1)$ $\quad 0\le t\le 2 \quad r'(t) = \langle -1,0\rangle,\ N = \langle 0, -1\rangle$

$$\int_{C_1} F\cdot N\,dt = \int_{-1}^{1} \langle x,y\rangle\cdot\langle -1,0\rangle\,dt = \int_{-1}^{1} 1\,dt = 2$$

$$\int_{C_2} F\cdot N\,dt = \int_{-1}^{1} \langle x,y\rangle\cdot\langle 0,1\rangle = \int_{-1}^{1} 1\,dt = 2$$

$$\int_{C_3} F\cdot N\,dt = \int_{-1}^{1} \langle x,y\rangle\cdot\langle 1,0\rangle\,dt = \int_{-1}^{1} 1\,dt = 2$$

$$\int_{C_4} F\cdot N\,dt = \int_{-1}^{1} \langle x,y\rangle\cdot\langle 0,-1\rangle\,dt = \int_{-1}^{1} 1\,dt = 2$$

So the flux is 8.

Conversely, by the flux form of Green's Theorem,

$$\text{Flux} = \oint_C F\cdot N\,dt = \int_{-1}^{1}\int_{-1}^{1} \operatorname{div} F\,dx\,dy = \int_{-1}^{1}\int_{-1}^{1} 2\,dx\,dy$$

$$= 8.$$

Ex: Let C be the Δ-path traversing $(0,0)$, $(1,0)$, and $(0,3)$ in that order. Calculate the flux of $F = \langle x^2 + e^y, x+y \rangle$ across S.

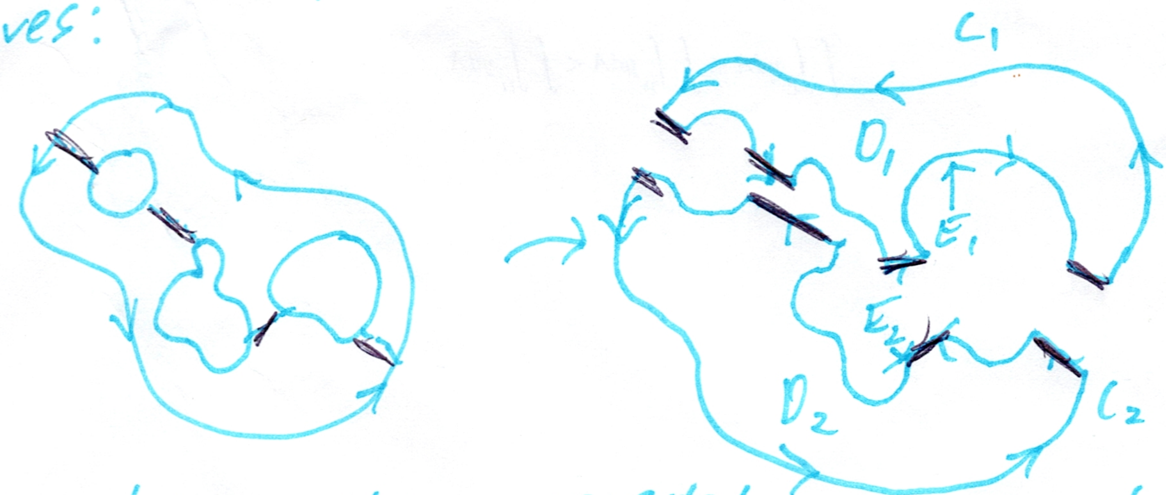
Sol: Note that this is negatively oriented. Thus $\oint_C F \cdot N \, dt = -\iint_D 2x+1 \, dA$, where D is $0 \leq x \leq 3$, $0 \leq y \leq 1 - \frac{x}{3}$. That is,

$-\int_0^3 \int_0^{1-x/3} 2x+1 \, dy \, dx = -\int_0^3 (2x+1)(1-\frac{x}{3}) dx =$

$-\int_0^3 2x - \frac{2x^2}{3} + 1 - \frac{x}{3} dx = -\int_0^3 1 + \frac{5}{3}x - \frac{2x^2}{3} dx =$

$-(x + \frac{5}{6}x^2 - \frac{2x^3}{9})\Big|_0^3 = -(3 + \frac{15}{2} - 6) = -(\frac{15}{2} - \frac{6}{2}) = -\frac{9}{2}$.

We stated Green's Theorem for shapes where the boundary is a simple, closed curve. However we can see that it works for shapes whose boundary are made up of multiple simple closed curves:



See how the repeated curves in the middle cancel out, so

$\oint_{C_1} F \cdot dr + \cancel{\oint_{E_1} F \cdot dr} + \cancel{\oint_{E_2} F \cdot dr} + \int_{C_2} F \cdot dr = \iint_{D_1} \text{curl} F \, dA + \iint_{D_2} \text{curl} F \, dA = \iint_D \text{curl} F \, dA$

$\oint F \cdot dr$