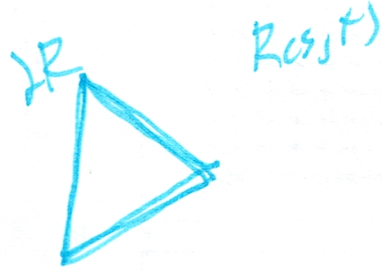
11/08 - Stokes Theorem

If $R(s,t)$ is a parametrized surface with parameter domain D, then the boundary of R, denoted ∂R, is the collection of all points on the edge of the surface.

Ex:



∂R is empty!

If our parametrization R is injective, then ∂R is the image of the boundary of D.

The surface is closed if it has no boundary. For example, the sphere is closed, but the cylinder is not.

Stokes's Theorem:

If S is a piecewise smooth, positively oriented surface whose boundary ∂S is a simple closed curve, and F is continuously differentiable, then

$$\int_{\partial S} F \cdot dr = \iint_S \operatorname{curl} F \cdot dS$$

Stoke's Theorem is a special case of Green's Theorem, as a positively oriented surface means that the boundary is counter clockwise.

Ex: Let $S = z = 4 - x^2 - y^2, z \geq 0$ and let $F = \langle y, 2z, x^2 \rangle$.

Then $\partial S = r(t) = (2\cos t, 2\sin t, 0)$, so

$$\int_C F \cdot dr = \int_0^{2\pi} \langle y, 2z, x^2 \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle \, dt$$

$$= \int_0^{2\pi} \langle 2\sin t, 0, 4\cos^2 t \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle \, dt$$

$$= \int_0^{2\pi} -4\sin^2 t \, dt = -2t + \sin 2t \Big|_0^{2\pi} = -4\pi$$

On the other hand, $\operatorname{curl} F = \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y, & 2z, & x^2 \end{pmatrix} = \langle -2, -2x, -1 \rangle$

$$T_x = \langle 1, 0, -2x \rangle$$

$$T_y = \langle 0, 1, -2y \rangle$$

$$N = \langle 2x, 2x, 1 \rangle.$$

This is the upwards normal vector, so we use it.

Then we have $\iint_D \langle -2, -2x, -1 \rangle \cdot \langle 2x, 2y, 1 \rangle \, dA$ $= \iint_D -4x - 4xy - 1 \, dA$. D is the circle of radius 2, so this is

$$\int_0^{2\pi} \int_0^2 -4r^2\cos\theta - 4r^3\cos\theta\sin\theta - r \, dr \, d\theta = \int_0^{2\pi} \left(-\tfrac{4}{3} r^3 \cos\theta - r^4 \cos\theta\sin\theta - \tfrac{r^2}{2} \right) \Big|_0^2 \, d\theta = \int_0^{2\pi} -\tfrac{32}{3}\cos\theta - 16\cos\theta\sin\theta - 2 \, d\theta =$$

$$\tfrac{32}{3}\sin\theta + 8\cos^2\theta - 2\theta \Big|_0^{2\pi} = -4\pi.$$

Ex: Calculate $\iint_S \text{curl} F \cdot dS$ over the pictured surface, where

$F = \langle z, 2xy, x+y \rangle$.

(The base is a circle of radius 1 centered at the origin.)

Note that even parametrizing this surface is a challenge. Instead we can use Stokes' Theorem:

∂S is the circle of radius 1 in the xy-plane, so $r(t) = \langle \cos t, \sin t, 0 \rangle$, $0 \le t \le 2\pi$.

Then $\int_C F \cdot dr = \int_0^{2\pi} \langle 0, 2\sin t \cos t, \cos t + \sin t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle dt$

$= \int_0^{2\pi} 2 \sin t \cos^2 t \, dt = \frac{2}{3}(-\cos^3 t)\Big|_0^{2\pi} = 0.$

Note that if C is a simple closed curve, and R, S are two different surfaces with $\partial R = \partial S = C$, then

$$\iint_S \text{curl} F \cdot dS = \iint_R \text{curl} F \cdot dS$$

by Stokes' Theorem, as both are equal to $\int_C F \cdot dr$.