In particular, if $G = \text{curl } F$, then for any two surfaces R and S with $\partial R = \partial S$, $\iint_R G \cdot dS = \iint_S G \cdot dS$. In other words, integrating G over a surface depends only on the boundary, and $\iint_R G \cdot dS = 0$ for any closed surface R. This is analogous to conservative vector fields and line integrals.

Let's describe vector fields G such that $G = \text{curl}(F)$: If $F = \langle P, Q, R \rangle$ and $G = \langle A, B, C \rangle$, then $\text{curl}(F) = G$ means

$$\begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{pmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle = \langle A, B, C \rangle$$

so

$$A = R_y - Q_z,$$

$$B = P_z - R_x,$$

and

$$C = Q_x - P_y$$

Recall that we can check for conservativity by comparing mixed partials. So consider:

$$A_x = R_{yx} - Q_{zx}$$

$$B_y = P_{zy} - R_{xy}$$

$$C_z = Q_{xz} - P_{yz}$$

Notice that the mixed partials appear with opposite signs. So

$$A_x + B_y + C_z = R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz}$$
$$= 0 = \text{div } G.$$

In other words, $G = \text{curl } F$ for some vector field if and only if $\text{div } G = 0$.
We call such a vector field <u>incompressible</u>. Then the surface integral of any incompressible vector field depends only on the boundary, and in particular the flux of an incompressible vector field over a closed surface is 0.

Ex: Find the flux of $F = \langle y^2 + e^{z^2}, \sin(x^3 + e^z), 8x^2 + \ln(y) \rangle$ through the sphere of radius 3 centered at $(5, 2, 1)$.

Sol: 0, as $\text{div } F = 0$.

Ex: Let F be a constant vector field and let S be a closed surface, then the flux of F through S is 0.

Ex: Let S be the cylinder of radius 2 and height 4 with a top but no bottom, oriented outwards. What is the flux of $F = \langle y^2, z^2, 3x+2 \rangle$ through S?

Sol: We could find G such that $\text{curl}(G) = F$ since $\text{div} F = 0$, then use Stokes' theorem to write $\iint_S F \cdot dS = \int_C G \cdot dr$, where C is the circle of radius 2 in the xy-plane.

Alternatively, we can find the flux through any other surface with the same boundary.

Let $R(r,\theta) = (r\cos\theta, r\sin\theta, 0)$, $0 \le r \le 2$, $0 \le \theta \le 2\pi$.

$$T_r = (\cos\theta, \sin\theta, 0)$$

$$T_\theta = (-r\sin\theta, r\cos\theta, 0)$$

$$N = T_r \times T_\theta = \langle 0, 0, r \rangle$$

Thus $$\iint_R F \cdot dS = \int_0^{2\pi}\int_0^2 r(3x+2)\,dr\,d\theta = \int_0^{2\pi}\int_0^2 3r^2\cos\theta + 2r\,dr\,d\theta$$

$$= \int_0^{2\pi} (r^3\cos\theta + r^2)\Big|_0^2 d\theta = \int_0^{2\pi} 8\cos\theta + 4\,d\theta =$$

$$8\pi + 8\sin\theta\Big|_0^{2\pi} = 8\pi.$$