11/12 - Divergence Theorem

Recall that Stokes' Theorem is the general version of the circulation form of Green's Theorem:

Stokes: $\int_C F \cdot dr = \iint_S \operatorname{curl}(F) \cdot dS$,

Green's: $\int_C F \cdot dr = \iint_D \operatorname{curl}(F)\, dA$

Also recall that $\operatorname{div}(F) = 0$ meant that the flux of F through a surface only depends on the boundary.

Much like $\operatorname{curl}(F)$ measures how nonconservative a vector field is, and Stokes'/Green's Theorem "added up" the curl at all the points in our region,

The divergence theorem is a similar generalization for the flux form of Green's Theorem.

Divergence Theorem: If S is a piecewise smooth, closed surface oriented outwards enclosing E, and F is a continuously differentiable vector field, then

$$\iiint_E \operatorname{div} F \, dV = \iint_S F \cdot dS$$

Note that this verifies the flux of any incompressible vector field through a closed surface is 0.

Ex: Let S be the cone $x^2+y^2=z^2$; $0\le z\le 1$ with a top, oriented outwards. Calculate the flux of $F=\langle x-y, x+z, z-y\rangle$ through S.

Sol: By the divergence theorem, $\iint_S F\cdot dS = \iiint_E \operatorname{div} F\, dS$. $\operatorname{div} F = 1+0+1=2$, so the flux is $\int_0^{2\pi}\int_0^1\int_r^1 2r\,dz\,dr\,d\theta =$

$$\int_0^{2\pi}\int_0^1 2r-2r^2\,dr\,d\theta = \int_0^{2\pi} r^2-\frac{2r^3}{3}\Big|_0^1 d\theta =$$

$$\int_0^{2\pi} \tfrac{1}{3}\,d\theta = \frac{2\pi}{3}.$$

You can double check that your flux integral gives the same answer.

Ex: Calculate $\iint_S F\cdot dS$, where S is the closed cylinder $x^2+y^2=1$, $0\le z\le 2$, and

$$F=\langle \tfrac{x^3}{3}+yz, \tfrac{x^3}{3}-\sin(xz), z-x-y\rangle.$$

Sol: By the divergence theorem,

$$\iint_S F\cdot dS = \iiint_E \operatorname{div} F = \int_0^{2\pi}\int_0^1\int_0^2 r(x^2+y^2+1)\,dz\,dr\,d\theta$$

$$= 2\int_0^{2\pi}\int_0^1 r^3+r\,dr\,d\theta = 2\int_0^{2\pi} \frac{r^4}{4}+\frac{r^2}{2}\Big|_0^1 d\theta =$$

$$2\int_0^{2\pi} \tfrac{3}{4}\,d\theta = 3\pi.$$

Just like with Green's Theorem, we can use the divergene theorem when the surface is not closed.

Ex: Let $F = \langle y^2, z^3, x^2 \rangle$. Compute the flux of F through the upper hemisphere of the unit sphere, oriented upwards.

Sol: The upper hemisphere of the unit sphere S plus the unit circle R is a closed surface. Thus by the divergence theorem, $\iint_R F \cdot dS + \iint_B F \cdot dS = \iiint_E \operatorname{div} F \cdot dS$. As $\operatorname{div} F = 0$, $\iint_R F \cdot dS = -\iint_S F \cdot dS$.

Then $R = \langle r\cos\theta, r\sin\theta, 0 \rangle$, $0 \le \theta \le 2\pi$, $0 \le r \le 1$.

$$T_r = \langle \cos\theta, \sin\theta, 0 \rangle$$

$$T_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$$

$$N = T_r \times T_\theta = \langle 0, 0, r \rangle$$

We need $\langle 0, 0, -r \rangle$ to make the surface $R+S$ positively oriented. Thus

$$\iint_B F \cdot ds = -\iint_R \langle y^2, z^3, x^2 \rangle \cdot \langle 0, 0, -r \rangle \, dA =$$

$$-\int_0^{2\pi}\int_0^1 -r^3\cos^2\theta \, dr \, d\theta = -\int_0^{2\pi} -\frac{r^4}{4}\cos^2\theta \Big|_0^1 d\theta =$$

$$\frac{1}{4}\int_0^{2\pi} \cos^2\theta \, d\theta = \frac{1}{4}\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Big|_0^{2\pi} =$$

$$\pi/4.$$