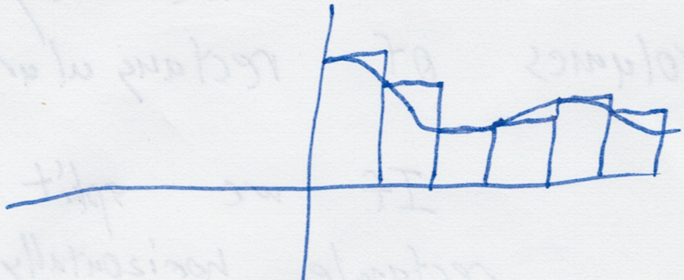
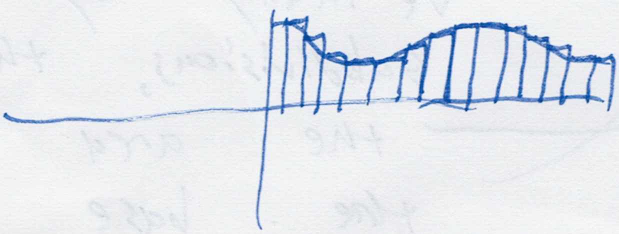
9/15 - Iterated Integrals

Recall Calc I: Given $f(x) \geq 0$, we used a Riemann sum to estimate the area under the curve of f:



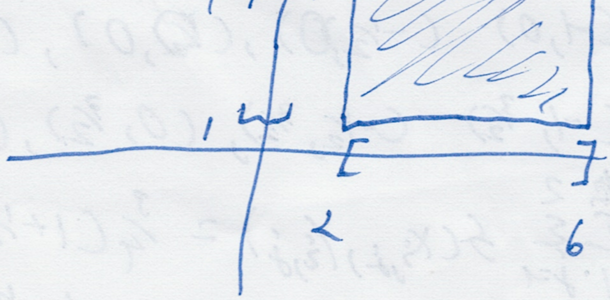
By taking smaller and smaller widths of our rectangles, we got closer and closer to the true area.



Taking the limit of this process gave us the integral.

We do the same for functions in three variables. Remember that $[a, b]$ is the inclusive interval between a and b.
$[a,b] \times [c,d]$ is all of the pairs (x,y) where x is in the first interval and y is in the second.

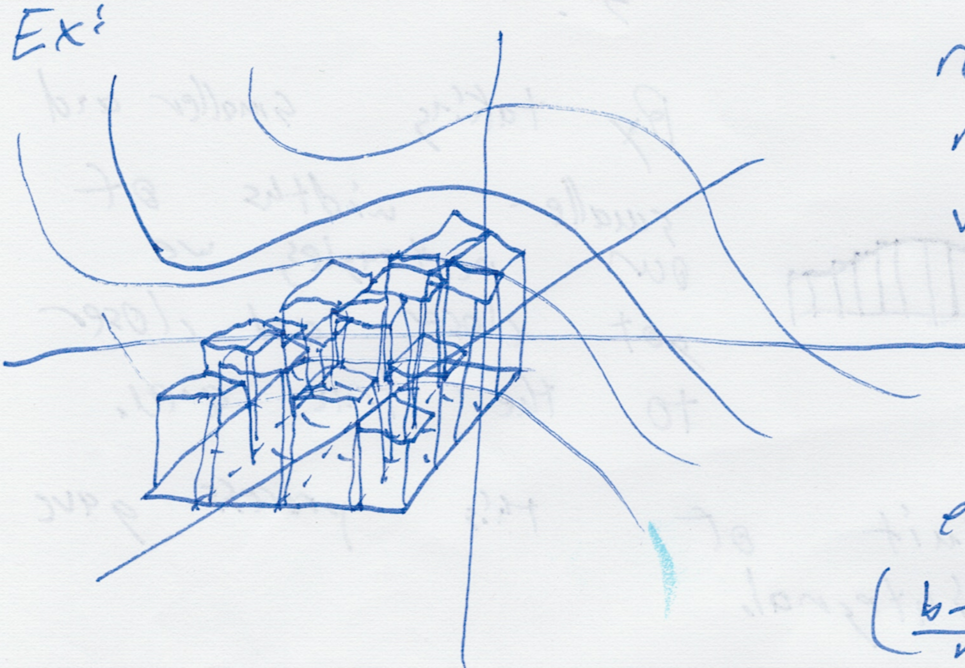
Ex:



$\leftarrow [2,6] \times [1,4]$

If we have some function $f(x,y) \geq 0$ defined on $[a,b] \times [c,d]$, then we estimate the volume beneath the surface by adding up the volumes of rectangular prisms.

Ex:

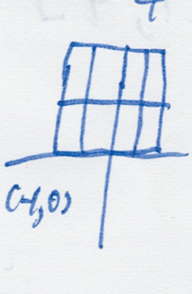


If we split the rectangle horizontally by n subdivisions and vertically by m subdivisions, then the area of the base of each rectangle is $\left(\frac{b-a}{n}\right)\left(\frac{d-c}{m}\right)$. The height is determined by the function value at some sample point inside the rectangle. This can be chosen however we want, i.e. lower left corner, midpoint, etc. We then add up all of them: $S_{n,m} = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{i,j}, y_{i,j}) \cdot \left(\frac{b-a}{n}\right)\left(\frac{d-c}{m}\right)$

Ex: $f(x,y) = x^2 + y^2$, $[-1,1] \times [0,3]$, $n=4$, $m=2$, bottom left

Sol: $\Delta A = \frac{2}{4} \cdot \frac{3}{2} = 3/4$ $\longrightarrow$ $(-1,0)$ $(-\frac{1}{2},0)$, $(0,0)$, $(\frac{1}{2},0)$

$(-1,\frac{3}{2})$ $(-\frac{1}{2},\frac{3}{2})$, $(0,\frac{3}{2})$, $(\frac{1}{2},\frac{3}{2})$



(-1,0)

$$S_{4,2} = \frac{3}{4} \sum_{i=1}^{4} \sum_{j=1}^{2} f(x_{i,j}, y_{i,j}) = \frac{3}{4}\left(1 + \frac{1}{4} + 0 + \frac{1}{4} + \frac{13}{4} + \frac{10}{4} + \frac{9}{4} + \frac{10}{4}\right)$$

$$= 12 \cdot \frac{3}{4} = 9.$$

As for in Calc 1, the volume is the limit of these approximations:

$$V = \lim_{n\to\infty} \lim_{m\to\infty} S_{m,n} = \frac{(b-a)}{n}\frac{(d-c)}{m}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{i,j}, y_{i,j})$$

We let $R = [a,b]\times[c,d]$. Then we define the double integral over R to be this limit:

$$\iint_R f(x,y)\,dA = \lim_{n\to\infty}\lim_{m\to\infty} S_{m,n}$$

This definition works for any function, not just $f(x,y)\geq 0$: it is then the "signed volume", not volume.

We say f is integrable over R if this exists. (As it is a limit, it may not.)

Ex: $f(x,y)=1$ if $x+y$ is rational, 0 otherwise.

dA can be written as $dx\,dy$, as $\boxed{dA}$ dy,
dx

<u>Properties of Double Integrals</u> (skipped on 9/15, will cover on 9/17)

Assume that $f(x,y)$ and $g(x,y)$ are integrable over R, S, T are rectangular subregions of R, and m, M are real numbers.

1. $\iint_R f(x,y)+g(x,y)\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$

2. $\iint_R m f(x,y)\,dA = m\iint_R f(x,y)\,dA$

3. If S and T only intersect on their boundary (or not at all), $\iint_{S\cup T} f(x,y)\,dA = \iint_S f(x,y)\,dA + \iint_T f(x,y)\,dA$

4. If $f(x,y) \geq g(x,y)$ on R,

$$\iint_R f(x,y)\, dA \geq \iint_R g(x,y)\, dA$$

5. If $m \leq f(x,y) \leq M$ on R, then

$$m\, A(R) \leq \iint_R f(x,y)\, dA \leq M A(R)$$

6. If $R = [a,b] \times [c,d]$ and $f(x,y) = h(x) g(y)$,

then $\iint_R f(x,y)\, dA = \left(\int_a^b h(x)\, dx\right)\left(\int_c^d g(y)\, dy\right)$.

Iterated Integrals

Fortunately, we can use what we already know about integrals to compute double integrals. An iterated integral is

$$\int_c^d \left(\int_a^b f(x,y)\, dx\right) dy \quad \text{or} \quad \int_a^b \left(\int_c^d f(x,y)\, dy\right) dx$$

The idea is that we fix one variable as a constant and then calculate the area under the curve of the resulting one-variable function. We then integrate over each of these to add up each area and find the volume.

Ex: $\int_0^3 \left(\int_4^8 y e^{xy}\, dx\right) dy$

sol: For any constant c, $\int e^{cx}\, dx = \frac{1}{c} e^{cx} + C$. As y is constant, $\int_4^8 y e^{xy}\, dx = e^{x}\Big|_4^8 = e^8 - e^4$. Then $\int_0^3 e^8 - e^4\, dy = (e^8 - e^4)\, y\Big|_0^3 = 3e^8 - 3e^4$.

Ex: $\int_{-3}^{2}\left(\int_{0}^{\pi/2} x^2\cos(yx)\,dy\right)dx$

Sol: $\int_{0}^{\pi/2} x^2\cos(yx)\,dy = (x\sin(yx))\Big|_0^{\pi/2} = x\sin(\frac{\pi x}{2})$.

$\int_{-3}^{2} x\sin(\frac{\pi x}{2})\,dx = \frac{-2x\cos(\frac{\pi x}{2})}{\pi}\Big|_{-3}^{2} + \int_{-3}^{2}\frac{2}{\pi}\cos(\frac{\pi x}{2})\,dx$

$= \frac{2}{\pi}\left(\frac{2}{\pi}\sin(\frac{\pi x}{2}) - x\cos(\frac{\pi x}{2})\right)\Big|_{-3}^{2}$

$= \frac{2}{\pi}\left(\frac{2}{\pi}\sin(\pi) - 2\cos(\pi) - \frac{2}{\pi}\sin(\frac{-3\pi}{2}) - 3\cos(\frac{3\pi}{2})\right)$

$= \frac{2}{\pi}\left(0 + 2 - \frac{2}{\pi} - 0\right) = \frac{4}{\pi} - \frac{4}{\pi^2}$

Fubini's Theorem V1:

If $R = [a,b]\times[c,d]$ and $f(x,y)$ is continuous over R, then

$$\iint_R f(x,y)\,dA = \int_c^d\int_a^b f(x,y)\,dx\,dy = \int_a^b\int_c^d f(x,y)\,dy\,dx$$

Ex: If $R = [-3,2]\times[0,\frac{\pi}{2}]$, compute $\iint_R x^2\cos(yx)\,dy\,dx$

Sol: $\iint_R x^2\cos(yx)\,dy\,dx = \int_{-3}^{2}\int_0^{\pi/2} x^2\cos(yx)\,dy\,dx = \frac{4}{\pi} - \frac{4}{\pi^2}$.

Ex: $\int_1^{16}\int_1^{8} \sqrt[4]{x} + 2\sqrt[3]{y}\,dy\,dx$

$\int_1^{16}\int_1^{8} \sqrt[4]{x} + 2\sqrt[3]{y}\,dy\,dx = \int_1^{16}\left(\sqrt[4]{x}\,y + \frac{3}{2}y^{4/3}\right)\Big|_1^8 dx =$

$\int_1^{16} \sqrt[4]{x}\cdot 8 + 24 - \sqrt[4]{x} - \frac{3}{2}\,dx = \int_1^{16} 7\sqrt[4]{x} + \frac{45}{2}\,dx = \left(\frac{28}{5}x^{5/4} + \frac{45}{2}x\right)\Big|_1^{16}$

$= \frac{32\cdot 28}{5} + 8\cdot 45 - \frac{28}{5} - \frac{45}{2}$

OR

$\int_1^{8}\int_1^{16} \sqrt[4]{x} + 2\sqrt[3]{y}\,dx\,dy = \int_1^8\left(\frac{4}{5}x^{5/4} + 2\sqrt[3]{y}\,x\right)\Big|_1^{16} dy =$

$\int_1^8 \frac{31\cdot 4}{5} + 30\sqrt[3]{y}\,dy = \left(\frac{31\cdot 4}{5}y + \frac{90}{4}y^{4/3}\right)\Big|_1^8 = \frac{31\cdot 28}{5} + 360 - \frac{90}{4}$

$= \frac{31\cdot 28}{5} + 8\cdot 45 - \frac{45}{2}$