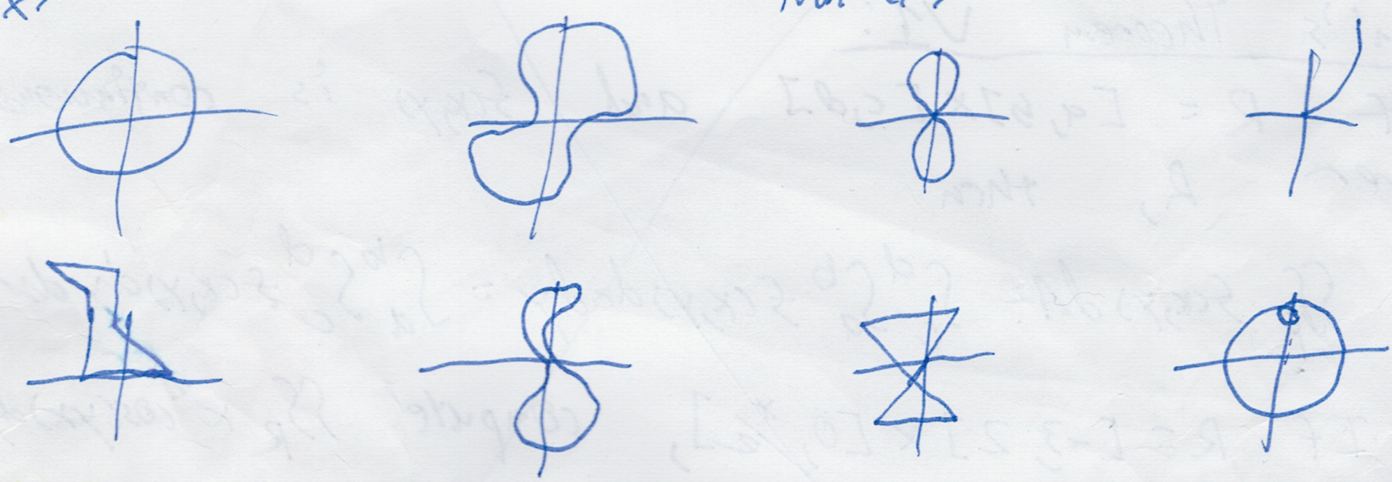
9/17 - ~~Iterated~~ Integrals over More General Regions

On Wednesday, we learned about double integrals over rectangular regions and using iterated integrals to compute them. Turns out, we can do this for more general regions.

A curve is simple and closed if it never intersects itself and starts and ends at the same point.

Ex: Non-ex:



We can integrate over regions which are bounded by simple closed curves. (We can break up into multiple regions if the curve is not simple.)

The formal definition takes the region D and puts it inside a rectangle R. Given a function f defined on D, define

$$\tilde{f}(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not } D. \end{cases}$$

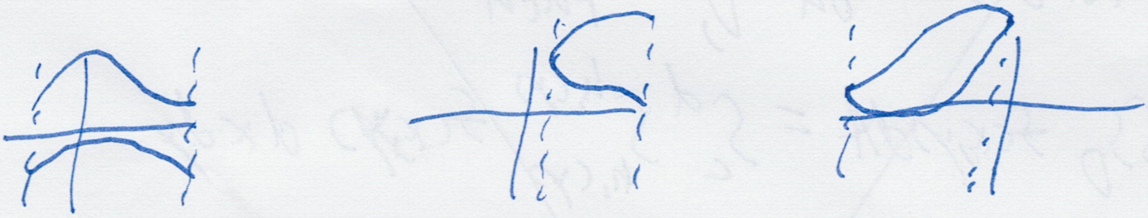
Then

$$\iint_D f(x,y)\,dA = \iint_R \tilde{f}(x,y)\,dA.$$

Much like the Riemann sum definition of double integrals, this is a technical definition we don't care about in practice.

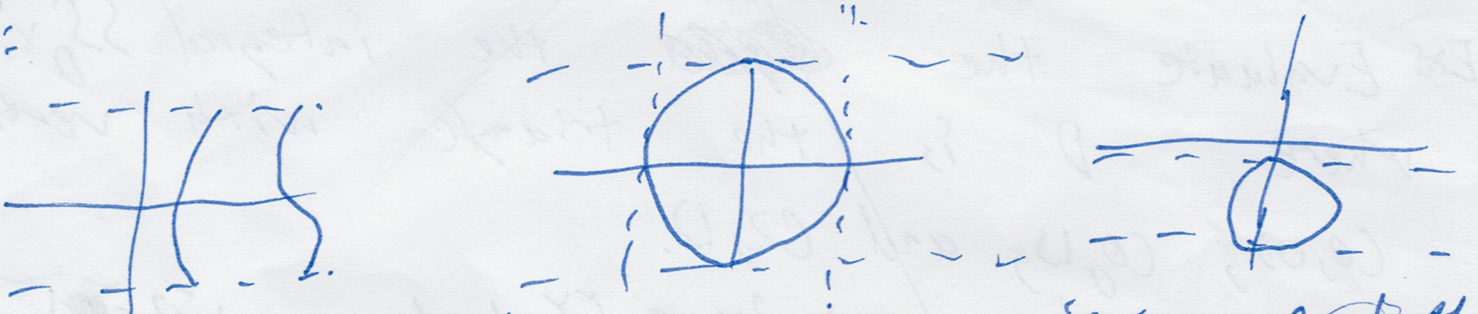
<u>Definition:</u> A region D is Type I if it lies between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$. I.e., $D = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

Ex:



Definition: A region D is Type II if it lies between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(y)$.
I.e., $D = \{(x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d\}$.

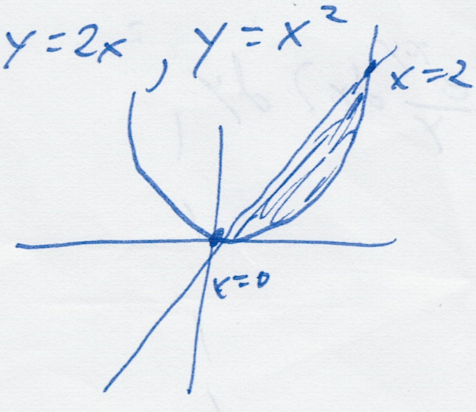
Ex:



As we see above, some regions are both

Ex: $y = 2x$, $y = x^2$

$0 \le x \le 2$, $x^2 \le y \le 2x$



$0 \le y \le 4$ $\quad \frac{y}{2} \le x \le \sqrt{y}$

$\frac{y}{2} = x$, $\sqrt{y} = x$

Fubini's Theorem V2

If D is a type I region and $f(x,y)$ is continuous on D, then

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

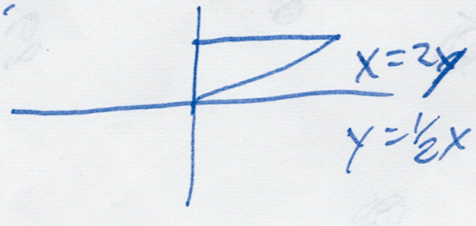
If D is a type II region and $f(x,y)$ is continuous on D, then

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

If our region is not both types, then it might be very difficult to switch the order of integration.

Ex: Integrate xe^{xy} over the region D, where D is the triangle with vertices $(0,0)$, $(0,1)$, and $(2,1)$

Sol:



$$\int_0^1 \int_0^{2y} x^2 e^{xy}\,dx\,dy =$$

$$\int_0^1 \left(\frac{x^2}{y} e^{xy}\Big|_0^{2y} - \int_0^{2y} 2x e^{xy}\,dx \right) dy = \text{Yuck}$$

$$\cancel{\int_0^1 \left(2e^{2y^2} - \frac{0}{y}e^0 - \frac{e^{2y^2}}{4} + \frac{1}{y} \right) dy =}$$

$$\cancel{\int_0^1 \frac{e^{2y^2}}{y} + \frac{1}{y}\,dy} \leftarrow \text{Stuck!}$$

$$\int_0^2 \int_{\frac{1}{2}x}^1 x^2 e^{xy}\,dy\,dx =$$

$$\int_0^2 \left(xe^{xy}\Big|_{\frac{1}{2}x}^1 \right) dx =$$

$$\int_0^2 xe^x - xe^{\frac{x^2}{2}}\,dx =$$

$$(xe^x)\Big|_0^2 - \int_0^2 e^x\,dx - \int_0^2 xe^{\frac{x^2}{2}}\,dx =$$

$$\left(xe^x - e^x - e^{\frac{x^2}{2}}\right)\Big|_0^2$$

$$2e^2 - e^2 - e^2 - 0 + e^0 + e^0 = 2.$$

Properties of Double Integrals

1. $\iint_D f(x,y) + g(x,y)\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$

2. $\iint_D m f(x,y)\,dA = m\iint_D f(x,y)\,dA$

3. If S and T only intersect on their boundaries (or not at all) and are subregions of D which together are all of D,

$$\iint_D f(x,y)\,dA = \iint_S f(x,y)\,dA + \iint_T f(x,y)\,dA$$

4. If $f(x,y) \geq g(x,y)$ on D,

$$\iint_D f(x,y)\,dA \geq \iint_D g(x,y)\,dA$$

5. If $m \leq f(x,y) \leq M$ on D,

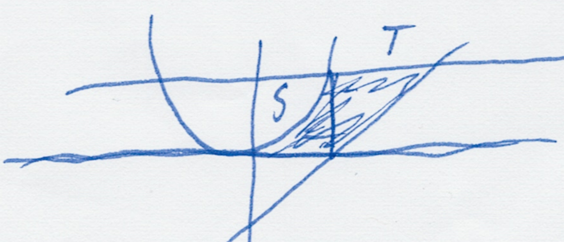
$$m\,\text{Area}(D) \leq \iint_D f(x,y)\,dA \leq M\,\text{Area}(D)$$

6. If $R = [a,b]\times[c,d]$ and $f(x,y) = h(x)g(y)$,

$$\iint_R f(x,y)\,dA = \left(\int_a^b h(x)\,dx\right)\left(\int_c^d g(y)\,dy\right)$$

#3 is especially important.

Ex: Let D be the region bounded by $y=1$, $y=0$, $y=x^2$, and $y=x-1$. Calculate $\iint_D x+y\,dA$



$$\iint_D x+y\,dA = \iint_S x+y\,dA + \iint_T x+y\,dA$$

$$= \int_0^1\int_0^{x^2} x+y\,dy\,dx + \int_0^1\int_1^{y+1} x+y\,dx\,dy$$

$$= \int_0^1 \left(xy+\frac{y^2}{2}\right)\Big|_0^{x^2} dx + \int_0^1 \left(\frac{x^2}{2}+xy\right)\Big|_1^{y+1} dy$$

$$= \int_0^1 x^3 + \frac{x^4}{2}\,dx + \int_0^1 \frac{3y^2}{2} + y \quad dy$$

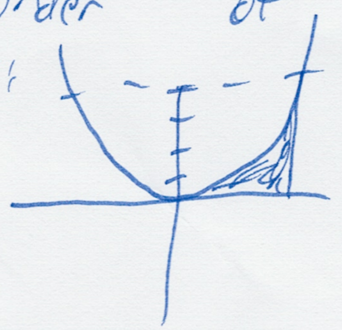
$$= \left(\frac{x^4}{4} + \frac{x^5}{10}\right)\Big|_0^1 + \left(\frac{y^3}{2} + \frac{y^2}{2}\right)\Big|_0^1$$

$$= \frac{1}{4} + \frac{1}{10} + \frac{1}{2} + \frac{1}{2} = \frac{27}{20}$$

As we have already seen, sometimes integrating is easier when we change the order of integration:

Ex: Evaluate $\int_0^4 \int_{\sqrt{y}}^2 \sin x^3 \, dx\,dy$ by changing the order of integration.

Sol:



$0 \le x \le 2$

$0 \le y \le x^2$

$$\int_0^2 \int_0^{x^2} \sin x^3 \, dy\,dx =$$

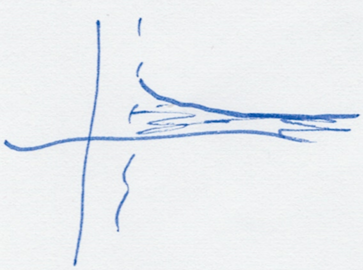
$$\int_0^2 x^2 \sin x^3 \, dx =$$

$$-\frac{\cos x^3}{3}\Big|_0^2 = \frac{-\cos 8}{3} + \frac{1}{3}$$

Improper integrals

Just like in single variable calculus, we can use limits to compute improper integrals.

Ex: Evaluate $\iint_D y^2 \, dA$, where $x \ge 1$, $0 \le y \le \frac{1}{x^2}$.



$$\lim_{c \to \infty} \int_1^c \int_0^{1/x^2} y^2 \, dy\,dx = \lim_{c \to \infty} \int_1^c \frac{y^3}{3}\Big|_0^{1/x^2} dx =$$

$$\lim_{c \to \infty} \int_1^c \frac{1}{3x^6} dx = \lim_{c \to \infty} -\frac{1}{15x^5}\Big|_1^c = \lim_{c \to \infty} \frac{1}{15} - \frac{1}{15c^5}$$

$$= \frac{1}{15}.$$