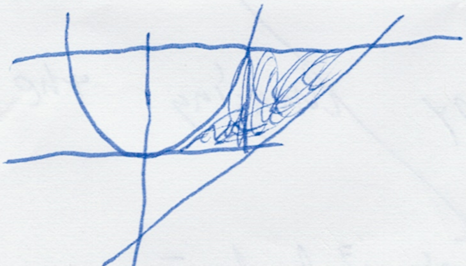
9/20 - Triple Integrals

Review of Double Integrals

Ex: Let D be the region bounded by $y=1$, $y=0$, $y=x^2$, and $y=x-1$. Calculate $\iint_D x+y\,dA$.

Sol: 

$0 \le x \le 1$, $0 \le y \le x^2$ and
$1 \le x \le 2$, $x-1 \le y \le 1$

or

$0 \le y \le 1$, $\sqrt{y} \le x \le y+1$

$\int_0^1 \int_0^{x^2} x+y\,dy\,dx + \int_1^2 \int_{x-1}^{1} x+y\,dy\,dx$; $\int_0^1 \int_{\sqrt{y}}^{y+1} x+y\,dx\,dy =$

$\int_0^1 xy + \frac{y^2}{2}\Big|_0^{x^2} dx + \int_1^2 xy + \frac{y^2}{2}\Big|_{x-1}^{1} dx$; $\int_0^1 \frac{x^2}{2} + xy\Big|_{\sqrt{y}}^{y+1} dy =$

$\int_0^1 x^3 + \frac{x^4}{2}\,dx + \int_1^2 x + \frac{1}{2} - x^2 + x - \frac{x^2}{2} + x - \frac{1}{2}\,dx$; $\int_0^1 \frac{y^2}{2} + y + \frac{1}{2} + y^2 + y - \frac{y}{2} - y^{3/2}\,dy$

$= \int_0^1 \frac{3}{2}y^2 + \frac{3}{2}y - y^{3/2} + \frac{1}{2}\,dy$

$\int_0^1 x^3 + \frac{x^4}{2} + \int_1^2 -\frac{3}{2}x^2 + 3x\,dx$; $= \left(\frac{y^3}{2} + \frac{3}{4}y^2 - \frac{2}{5}y^{5/2} + \frac{y}{2}\right)\Big|_0^1 =$

$\frac{x^4}{4} + \frac{x^5}{10}\Big|_0^1 + \left(-\frac{x^3}{2} + \frac{3x^2}{2}\right)\Big|_1^2$; $= \frac{1}{2} + \frac{3}{4} - \frac{2}{5} + \frac{1}{2} = \frac{27}{20}$

$= \frac{1}{4} + \frac{1}{10} - 4 + 6 + \frac{1}{2} - \frac{3}{2} =$

$2 + \frac{1}{4} + \frac{1}{10} = \frac{27}{20}$

We can do improper integrals using limits as in the one variable case.

Ex: Let D be the region defined via $x \ge 1$, $0 \le y \le \frac{1}{x^2}$. Compute $\iint_D y^2\,dA$.

Sol: $\iint_D y^2\,dA = \lim_{c\to\infty}\int_1^c\int_0^{1/x^2} y^2\,dy\,dx = \lim_{c\to\infty}\int_1^c \frac{y^3}{3}\Big|_0^{1/x^2}dx =$
$\lim_{c\to\infty}\int_1^c \frac{1}{3x^6}dx = \lim_{c\to\infty}\left(-\frac{1}{15x^5}\right)\Big|_1^c = \lim_{c\to\infty}\frac{1}{15} - \frac{1}{15c^5} = \frac{1}{15}.$

Triple Integrals

We can use Riemann sums on boxes $B=[a,b]\times[c,d]\times[e,f]$ to define triple integrals like we did double integrals. Like before, a triple integral is the limit of Riemann sums of "masses" (i.e. four dimensional volume) of boxes as the size of the boxes goes to zero.

As before, we won't use this definition for practical calculations.

Fubini's Theorem V3:

If $D=\{(x,y):$ There is some z with (x,y,z) in $E\}$

and $E=\{(x,y,z): (x,y)$ is in D and $u_1(x,y)\le z\le u_2(x,y)\}$,

Then $\iiint_E f(x,y,z)\,dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right)dA.$

The order of variables can change. If we have a region D in the yz plane and a function of x beside it, we can still do a triple integral. Same with the xz-plane and y.

Examples: Let E be the tetrahedron with vertices $(0,0,0)$, $(0,0,4)$, $(2,0,0)$, and $(0,1,0)$. Calculate $\iiint_E 3x\ dV$.

Sol: We need to describe E. The tetrahedron has each coordinate plane containing one face, and we need to find the plane containing the fourth. We use the cross product: $\langle -2,1,0\rangle$ and $\langle -2,0,4\rangle$ are in the plane.

$$\langle -2,1,0\rangle \times \langle -2,0,4\rangle = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 0 \\ -2 & 0 & 4 \end{bmatrix} = 4\hat{i} + 8\hat{j} + 2\hat{k}.$$

$4x+8y+2z=d$. As $(0,1,0)$ is in the plane, $4x+8y+2z=8$, or $2x+4y+z=4$.

Then z goes from 0 to $4-2x-4y$.

For the bounds on x and y, we look at the intersection with the xy-plane, i.e. $z=0$. Then $2x+4y=4$, or $y=1-\frac{x}{2}$. So the y bounds are from 0 to $1-\frac{x}{2}$. Finally, the largest x value is $x=2$.

So, $\int_0^2\int_0^{1-\frac{x}{2}}\int_0^{4-2x-4y} 3x+2y\,dz\,dy\,dx =$

$\int_0^2\int_0^{1-\frac{x}{2}} (3xz+2yz)\Big|_0^{4-2x-4y} dy\,dx = \int_0^2\int_0^{1-\frac{x}{2}} 12x-6x^2-$

$+8y-4xy-8y^2\,dy\,dx = \int_0^2 (12xy-6x^2y- +4y^2$

$-\frac{2xy^2}{2}-\frac{8y^3}{3})\Big|_0^{1-\frac{x}{2}} dx = \int_0^2 12x-6x^2-6x^2+3x^3-\frac{3}{2}x+\frac{3}{2}x^2-$

$\frac{3}{8}x^3+4-4x-x^2-2x+2x^2-\frac{x^3}{2}-\frac{8}{3}+4x-2x^2+x^3\,dx =$

$\int_0^2 \frac{10}{3}x^3$ $+\frac{4}{3}$