Examples: Let E be the tetrahedron with vertices $(0,0,0)$, $(0,0,4)$, $(2,0,0)$, and $(0,1,0)$. Calculate $\iiint_E 3x$ [illegible] dV.

Sol: We need to describe E. The tetrahedron has each coordinate plane containing one face, and we need to find the plane containing the fourth. We use the cross product: $\langle -2,1,0\rangle$ and $\langle -2,0,4\rangle$ are in the plane.

$$\langle -2,1,0\rangle \times \langle -2,0,4\rangle = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 0 \\ -2 & 0 & 4 \end{bmatrix} = 4\hat{i} + 8\hat{j} + 2\hat{k}.$$

$4x+8y+2z=d$. As $(0,1,0)$ is in the plane, $4x+8y+2z=8$, or $2x+4y+z=4$.

Then z goes from 0 to $4-2x-4y$. For the bounds on x and y, we look at the intersection with the xy-plane, i.e. $z=0$. Then $2x+4y=4$, or $y=1-\frac{x}{2}$. So the y bounds are from 0 to $1-\frac{x}{2}$. Finally, the largest x value is $x=2$.

So, $\int_0^2 \int_0^{1-\frac{x}{2}} \int_0^{4-2x-4y} 3x+2y \, dz\,dy\,dx =$

$\int_0^2 \int_0^{1-\frac{x}{2}} (3xz+2yz)\Big|_0^{4-2x-4y} dy\,dx = \int_0^2 \int_0^{1-\frac{x}{2}} 12x - 6x^2 - [\text{illegible}] + 8y - 4xy - 8y^2 \, dy\,dx = \int_0^2 (12xy - 6x^2y - [\text{illegible}] + 4y^2 - 2xy^2 - \frac{8y^3}{3})\Big|_0^{1-x/2} dx = \int_0^2 12x - 6x^2 - 6x^2 + 3x^3 - \frac{3}{2}x + \frac{3}{2}x^2 - \frac{3}{8}x^3 + 4 - 4x - x^2 - 2x + 2x^2 - \frac{x^3}{2} - \frac{8}{3} + 4x - 2x^2 + x^3 \, dx =$

$+\frac{4}{3}$

$\int_0^2 \frac{10}{3}x^3$

So, $\int_0^2 \int_0^{1-\frac{x}{2}} \int_0^{4-2x-4y} 3x \, dz\, dy\, dx = \int_0^2 \int_0^{1-\frac{x}{2}} 3xz\Big|_0^{4-2x-4y} dy\, dx =$

$\int_0^2 \int_0^{1-\frac{x}{2}} 12x - 6x^2 - 12xy \, dy\, dx = \int_0^2 (12xy - 6x^2y - 6xy^2)\Big|_0^{1-\frac{x}{2}} dx$

$= \int_0^2 12x - 6x^2 - 6x^2 + 3x^3 - 6x + 6x^2 - \frac{3}{2}x^3 \, dx =$

$\int_0^2 \frac{3}{2}x^3 - 6x^2 + 6x \, dx = (\frac{3}{8}x^4 - 2x^3 + 3x^2)\Big|_0^2 = 6 - 16 + 12 = 2.$

We can integrate in a different order: Find the volume of the above tetrahedron.

Sol: Finding area/volume is done by integrating the constant 1 function over the region. ~~The largest z value is 4.~~ We solve for y in terms of x and z: $y = 1 - \frac{x}{2} - \frac{z}{4}$. When $y=0$, $x = 2 - \frac{z}{2}$. Finally, z's largest value is 4. Thus the volume is given by

$$\int_0^4 \int_0^{2-\frac{z}{2}} \int_0^{1-\frac{x}{2}-\frac{z}{4}} 1 \, dy\, dx\, dz = \int_0^4 \int_0^{2-\frac{z}{2}} 1 - \frac{x}{2} - \frac{z}{4} \, dx\, dz =$$

$$\int_0^4 (1-\frac{z}{4})x - \frac{x^2}{4}\Big|_0^{2-\frac{z}{2}} dz = \int_0^4 (1-\frac{z}{4})(2-\frac{z}{2}) - \frac{(2-\frac{z}{2})^2}{4} dz =$$

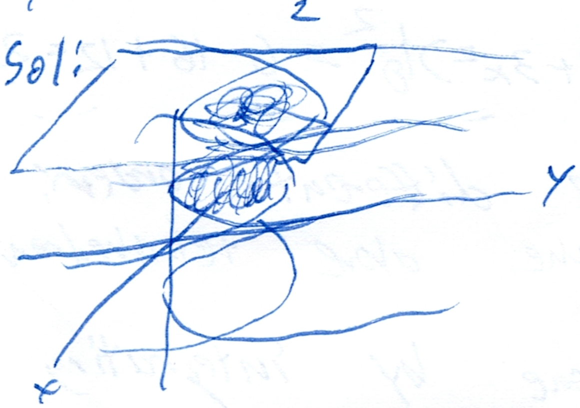
$$\int_0^4 2 - z + \frac{z^2}{8} - 1 + \frac{z}{2} - \frac{z^2}{16} \, dz = \int_0^4 1 - \frac{z}{2} + \frac{z^2}{16} \, dz =$$

$$(z - \frac{z^2}{4} + \frac{z^3}{48})\Big|_0^4 = 4 - 4 + \frac{64}{48} = \frac{4}{3}.$$

9/22 - Triple Integrals 2

Ex: Calculate $\iiint_E x^2\,dV$, where E is the region bounded by $z=0$, $z=1$, $y=x^2$, and $y=1-x^2$.

Sol:



We need to find where $y=x^2$ meets $y=1-x^2$.

$1-x^2=x^2$, or $1=2x^2$, $x=\pm\frac{1}{\sqrt2}$

Therefore, our x bounds are from $-\frac{1}{\sqrt2}$ to $\frac{1}{\sqrt2}$, and our y bounds are from x^2 to $1-x^2$.

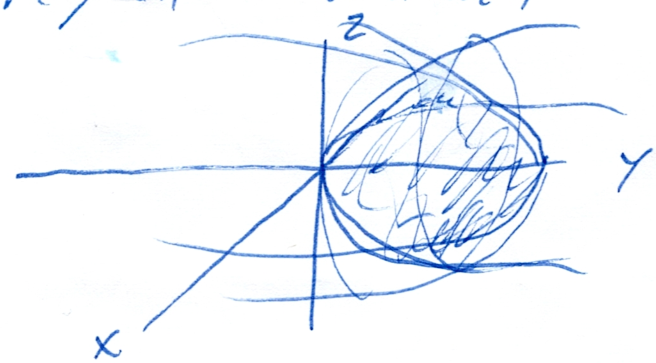
$$\int_0^1\int_{-1/\sqrt2}^{1/\sqrt2}\int_{x^2}^{1-x^2} x^2\,dy\,dx\,dz = \int_0^1\int_{-1/\sqrt2}^{1/\sqrt2} x^2-x^4-x^4\,dx\,dz = \int_0^1\int_{-1/\sqrt2}^{1/\sqrt2} x^2-2x^4\,dx\,dz$$

$$=\int_0^1 \frac{x^3}{3}-\frac{2x^5}{5}\Big|_{-1/\sqrt2}^{1/\sqrt2} = \int_0^1 \frac{1}{6\sqrt2}-\frac{1}{10\sqrt2}+\frac{1}{6\sqrt2}-\frac{1}{10\sqrt2}\,dz =$$

$$\int_0^1 \frac{1}{3\sqrt2}-\frac{1}{5\sqrt2}\,dz = \frac{1}{3\sqrt2}-\frac{1}{5\sqrt2}.$$

In this example, the region really only depends on x and y, so we have two integrals with constant bounds.

Ex: Calculate $\iiint_E z\,dV$, where E is the region bounded by $y=x^2+z^2$ and $y=2-x^2-z^2$.



The xz-bounds are where these meet, i.e.

$x^2+z^2=2-x^2-z^2$, or

$x^2+z^2=1$.

Thus the z bounds are from -1 to 1, and the x bounds are from $-\sqrt{1-z^2}$ to $\sqrt{1-z^2}$. Therefore,

$$\iiint_E z\,dV = \int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{x^2+z^2}^{2-x^2-z^2} z\,dy\,dx\,dz =$$

$$\int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} z(2-2x^2-2z^2)\,dx\,dz = \int_{-1}^{1} z\left(2x-2xz^2-\frac{2x^3}{3}\right)\Big|_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} dz$$

$$= \int_{-1}^{1} z\left(2-2z^2-\tfrac{2}{3}(1-z^2)\right)\left(\sqrt{1-z^2}\right)dz = \int_{-1}^{1} \tfrac{4}{3}z(1-z^2)\sqrt{1-z^2}\,dz$$

$$= \tfrac{4}{3}\int_{-1}^{1} z(1-z^2)^{3/2}\,dz = -\tfrac{2}{3}\int_{-1}^{1}(1-z^2)^{3/2}(-2z\,dz) = -\tfrac{2}{3}\left(\frac{2(1-z^2)^{5/2}}{5}\right)\Big|_{-1}^{1}$$

$$=0.$$

~~Sometimes it is usually easiest with triple integrals to~~

we can split up regions of integration just like we did for double integrals.

Ex: Find the volume of the region bounded by $z=1-x^2-y^2$ and $z+1=\sqrt{x^2+y^2}$

We could integrate this as

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}-1}^{1-x^2-y^2} 1\,dV.$$

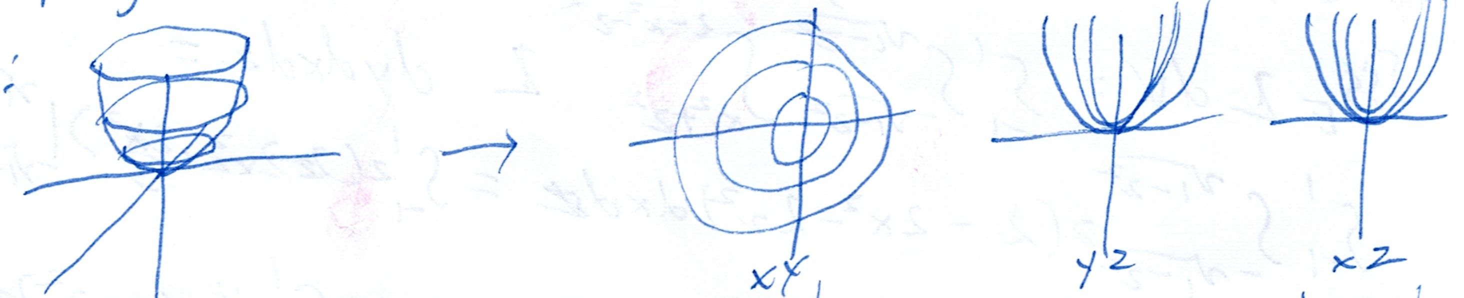
However, this looks hard.

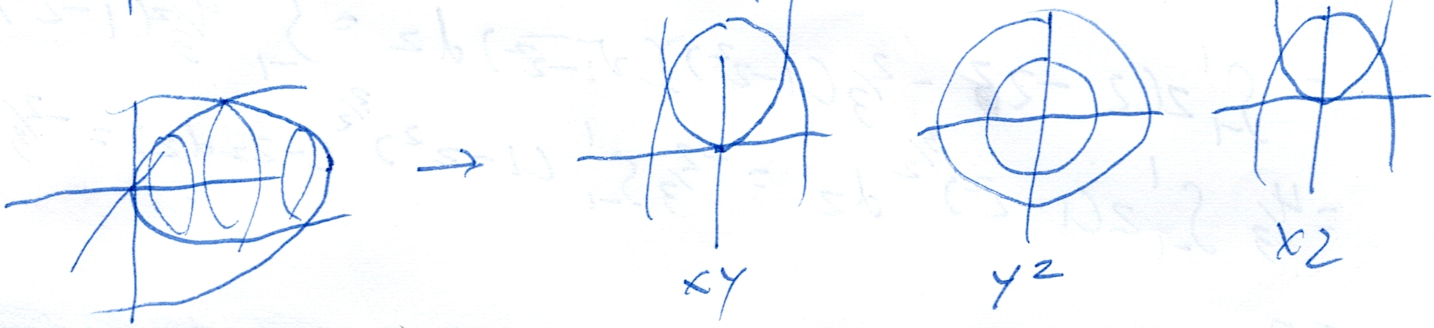
Instead, we can split it at $z=0$. The top half is half a sphere of radius 1, so has volume $\frac{2}{3}\pi$. The bottom half is a cone with radius and height 1, so it has volume $\pi/3$. Thus the total volume is π.

Tips for drawing pictures:

- Project onto the coordinate planes.

Ex:





- Find an axis of symmetry: Something like the above paraboloid is similar to Calc 1/2 rotation/washer problems. (This only works if your function of integration is also symmetric.)

- For homework/webwork, use an online grapher like Geogebra to help improve your intuition for what different functions look like.