## 9/24 - Polar coordinates

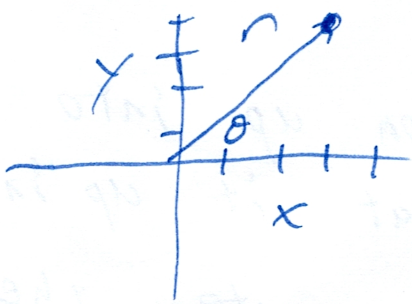
If we want to write a treasure map, there are two ways to give instructions: By North-South and East-west, or direction and distance.

I.e.: "3 steps west, ~~two~~ 4 steps North"

or

"Face north-west, 5 steps"

The same is true in the xy-plane: We can go over x and up y, or we can turn by angle $\theta$ from the positive x-axis then travel distance r in that direction.




Given a point $(x, y)$, we convert to polar coordinates $(r, \theta)$ via $r = \sqrt{x^2+y^2}$ and $\tan\theta = \frac{y}{x}$, with $0 \le \theta \le 2\pi$.

Ex:

| Cartesian | Polar |
|---|---|
| $(6,0)$ | $(6,0)$ |
| $(0,3)$ | $(3, \frac{\pi}{2})$ |
| $(2,2)$ | $(2\sqrt{2}, \frac{\pi}{4})$ |
| $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$ | $(1, \frac{4\pi}{3})$ |

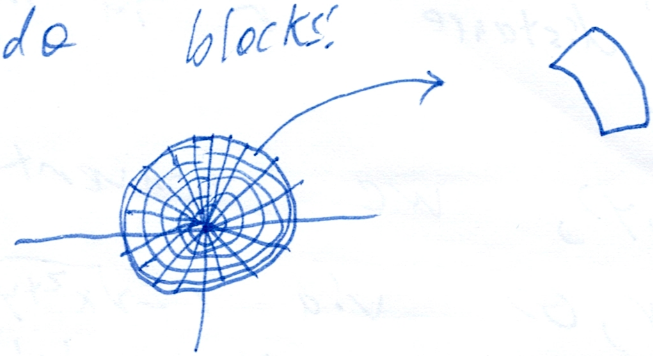
Polar coordinates define curves via functions just like Cartesian coordinates.

$r=\theta$ $\qquad$ $r=3\sin 2\theta$ $\qquad$ $r=4$

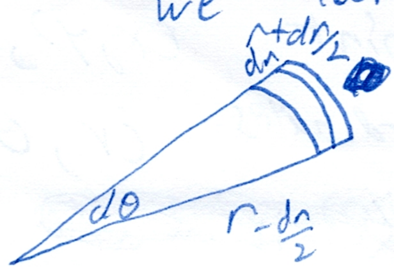


Some regions are much easier to describe in Polar coordinates, making them easier to integrate over as well.

Instead of cutting our region up into small $dxdy$ blocks, we cut it up into $drd\theta$ blocks:



To compute the area of this block, we look at




The area of an arc is $\frac{r^2\theta}{2}$ when $\theta$ is in radians, so

In the above picture the area of the block is $\frac{(r+\frac{dr}{2})^2\theta}{2} - \frac{(r-\frac{dr}{2})^2\theta}{2} = (r^2+rdr+\frac{dr^2}{4}-r^2+rdr-\frac{dr^2}{4})\frac{\theta}{2}$

$= rdrd\theta$

This is the main difference between polar and Cartesian coordinates! In the former, $dA = rdrd\theta$, NOT $drd\theta$

We can integrate over polar rectangular regions:

$\pi/4 \leq \theta \leq \pi/2$, $1 \leq r \leq 3$, $f(r,\theta) = \frac{1}{r}$

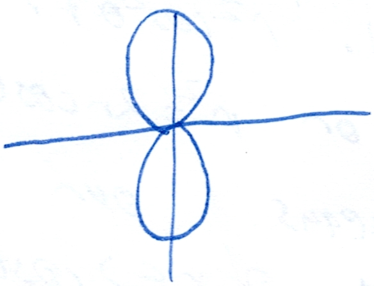
$$\int_{\pi/4}^{\pi/2}\int_1^3 \frac{1}{r}\, r\,dr\,d\theta = \int_{\pi/4}^{\pi/2}\int_1^3 dr\,d\theta =$$

$$\int_{\pi/4}^{\pi/2} 2\,d\theta = \pi/2.$$

As expected, we can integrate over more general regions too:

Ex: Integrate $x$ over the region $r \leq 3\sin\theta$

$x = r\cos\theta$



$$\int_0^{2\pi}\int_0^{3\sin\theta} r^2\cos\theta\,dr\,d\theta =$$

$$\int_0^{2\pi} \frac{r^3}{3}\cos\theta\Big|_0^{3\sin\theta} d\theta =$$

$$\int_0^{2\pi} 9\sin^3\theta\cos\theta\,d\theta = \frac{9\sin^4\theta}{4}\Big|_0^{2\pi} = 0.$$

We could switch the order in theory, but rarely is there a region for which it makes sense to do so. (Generally, it will involve inverse trig functions.)

Ex: $r \leq 1$, $\theta \leq \pi r$.

$$\int_0^1\int_0^{\pi r} 1\, r\,d\theta\,dr =$$

$$\int_0^1 \pi r^2\,dr = \frac{\pi r^3}{3}\Big|_0^1 = \frac{\pi}{3}.$$