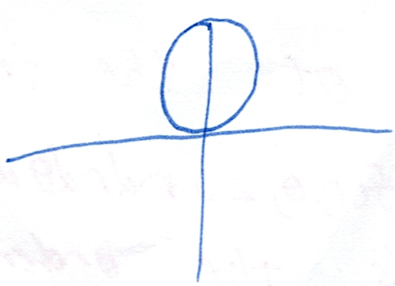
As with double integrals in Cartesian coordinates, double integrals in polar coordinates represent volumes.

Ex: $\int_0^{2\pi}\int_0^1 \sqrt{1-r^2}\, r\,dr\,d\theta = \int_0^{2\pi} -\frac{(1-r^2)^{3/2}}{3}\Big|_0^1 d\theta = \int_0^{2\pi} \frac{1}{3}\, d\theta =$ $\frac{2\pi}{3}$, which is the volume of the upper half of the sphere with radius 1.

Ex: Find the volume below $z=4-x^2-y^2$ and above the circle $x^2+(y-1)^2=1$.

Sol: $z=4-r^2$ is $z=4-x^2-y^2$ in polar coordinates



$x^2+(y-1)^2=1$ becomes $r^2\cos^2\theta+(r\sin\theta-1)^2=1$, which simplifies to $r^2\cos^2\theta+r^2\sin^2\theta-2r\sin\theta+1$, or $r^2-2r\sin\theta$. In other words, $r^2=2r\sin\theta$. Thus $r=0$ or $r=2\sin\theta$. From the picture, we see our $\theta$ bounds are from $0$ to $\pi$. (Also observe that $r$ is always positive, so $2\sin\theta$ must be. Open Stax makes a "mistake" here.)

$\int_0^{\pi}\int_0^{2\sin\theta}(4-r^2)\,r\,dr\,d\theta = \int_0^{\pi}\int_0^{2\sin\theta} 4r-r^3\,dr\,d\theta = \int_0^{\pi}\left(2r^2-\frac{r^4}{4}\right)\Big|_0^{2\sin\theta} d\theta$

$= \int_0^{\pi} 8\sin^2\theta-4\sin^4\theta\,d\theta = \int_0^{\pi} 4\sin^2\theta(2-\sin^2\theta)\,d\theta = \int_0^{\pi} 4\sin^2\theta(1+\cos^2\theta)$

$= \int_0^{\pi} 4\sin^2\theta+4(\sin\theta\cos\theta)^2\,d\theta = \int_0^{\pi} 4\sin^2\theta+\sin^2 2\theta\,d\theta =$

$(2x-\sin 2x)\Big|_0^{\pi}+\left(\frac{x}{2}-\frac{\sin 4x}{8}\right)\Big|_0^{\pi} = 2\pi+\frac{\pi}{2}=\frac{5\pi}{2}.$

9/27 - Cylindrical and Spherical coordinates

Recall that $\iiint f(x,y,z)\,dV$ was defined as $\iint_D \left(\int f(x,y,z)\,dz\right) dA$. Before, $\iint_D dA$ was always done in Cartesian coordinates, but nothing is stopping us from using polar coordinates!

Cylindrical Coordinates: Three dimensional coordinate system where $x,y$ is replaced by polar coordinates $(r,\theta)$.

Then our integral is of the form

$$\iiint f(x,y,z)\,dV = \iiint f(r,\theta,z)\, r\,dr\,d\theta\,dz.$$

As with other triple integrals, the order can be switched around.

Ex: Calculate the volume of a cylinder of radius 3 and height 4.

Sol: $\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^4 1\,dz\,dy\,dx$

$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 4\,dy\,dx$

$= \int_{-3}^{3} 8\sqrt{9-x^2}\,dx$

= Trig substitution, disgusting

$\int_0^{2\pi}\int_0^3\int_0^4 r\,dz\,dr\,d\theta$

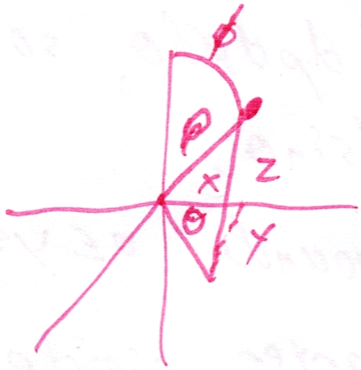
$= \int_0^{2\pi}\int_0^3 4r\,dr\,d\theta$

$= \int_0^{2\pi} 2r^2\Big|_0^3\,d\theta$

$= \int_0^{2\pi} 18\,d\theta = 36\pi$

$(\pi r^2 h)$

Spherical coordinates are like polar coordinates but for three dimensions.



To translate from spherical to polar coordinates, we use $(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$.

To go the other way,

$\rho^2 = x^2+y^2+z^2$, $\tan\theta = \frac{y}{x}$, $\cos\phi = \frac{z}{\sqrt{x^2+y^2+z^2}}$.

We can also convert between cylindrical and spherical coordinates.

$r = \rho\sin\phi$ $\quad$ $\rho = \sqrt{r^2+z^2}$

$\theta = \theta$ $\quad$ $\theta = \theta$

$z = \rho\cos\phi$, $\quad$ $\cos\phi = \frac{z}{\sqrt{r^2+z^2}}$.

When integrating with respect to spherical coordinates,

$dV = \rho^2\sin\phi\, d\rho\, d\theta\, d\phi$.

Ex: Find the volume of the part of the sphere of radius 2 between $\phi = \pi/3$ and $\phi = \frac{2\pi}{3}$.

Sol:

$$\int_0^{2\pi}\int_0^2\int_{\pi/3}^{2\pi/3} 1\,\rho^2\cdot\sin\phi\, d\phi\, d\rho\, d\theta =$$

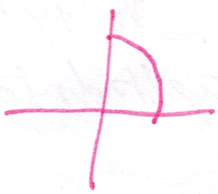
$$\int_0^{2\pi}\int_0^2 -\cos\phi\,\rho^2\Big|_{\pi/3}^{2\pi/3} d\rho\, d\theta =$$

$$\int_0^{2\pi}\int_0^2 \rho^2\, d\rho\, d\theta = \int_0^{2\pi} \frac{\rho^3}{3}\Big|_0^2 d\theta = \int_0^{2\pi}\frac{8}{3}\, d\theta = \frac{16}{3}\pi$$

Ex: Convert $\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\,dz\,dx\,dy$ into spherical coordinates.

Sol: $x^2+y^2+z^2 = \rho^2$, and $dz\,dx\,dy = \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$, so we are integrating $\rho^4 \sin\theta$.

Note that the bounds $0 \le y \le 3$, $0 \le x \le \sqrt{9-y^2}$, define the quarter circle



in the $xy$-plane. Thus $0 \le \theta \le \pi/2$.

For the lower bound on $z$, we have $z = \sqrt{x^2+y^2}$, or $z = r = \rho \sin\phi$. $z = \rho\cos\phi$, so we have $\rho\cos\phi = \rho\sin\phi$. Thus $\cos\phi = \sin\phi$, or $\phi = \pi/4$. Therefore $\phi$ ranges from $0$ to $\frac{\pi}{4}$.

Finally, note that $z \le \sqrt{18-x^2-y^2}$ implies $z^2 \le 18-x^2-y^2$, or $\rho^2 \le 18$. Thus $0 \le \rho \le 3\sqrt{2}$ are our $\rho$ bounds.