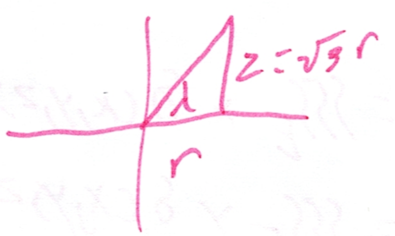
Ex: Find the mass of the solid above $z = \sqrt{3x^2+3y^2}$ and below $z = \sqrt{9-x^2-y^2}$ with density $\delta(x,y,z) = \sqrt{x^2+y^2+z^2}$

Sol:

If we convert this to spherical coordinates, we get:

$$z^2 = 9 - x^2 - y^2$$
$$x^2+y^2+z^2 = 9$$
$$\rho = 3.$$

Also, $z = \sqrt{3r^2} = \sqrt{3}\,r$.



Thus $\lambda = \pi/3$ since $\tan\lambda\, r = z$, so $\tan\lambda = \sqrt{3}$.

Therefore we have a spherical integral with bounds $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi/6$ (since $\pi/6$ is the complementary angle of $\pi/3$), and $0 \le \rho \le 3$. So the mass is given by

$$\int_0^{2\pi}\int_0^{\pi/6}\int_0^3 \rho \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/6}\int_0^3 \rho^3 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/6} \frac{\rho^4}{4}\sin\phi \Big|_0^3 \, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/6} \frac{81}{4}\sin\phi \, d\phi\, d\theta =$$

$$\int_0^{2\pi} \left(-\frac{81}{4}\cos\theta\right)\Big|_0^{\pi/6} d\theta = \int_0^{2\pi} \frac{81}{4} - \frac{81\sqrt{3}}{8}\, d\theta = \frac{81\pi}{2} - \frac{81\sqrt{3}\pi}{4}.$$

A <u>lamina</u> is a 2d shape D with mass. If $\delta(x,y)$ is the density of D at (x,y), then the mass is $\iint_D \delta(x,y)\,dA$.

If a 3d-shape E has density function $\delta(x,y,z)$, then its total mass is given by $\iiint_E \delta(x,y,z)\,dV$.

The moment of a lamina (or solid) about the x and y (and z) axis are given by

$$M_x = \iint_D x\,\delta(x,y)\,dA \qquad M_x = \iiint_E x\,\delta(x,y,z)\,dV$$

$$M_y = \iint_D y\,\delta(x,y)\,dA \qquad M_y = \iiint_E y\,\delta(x,y,z)\,dV$$

$$M_z = \iiint_E z\,\delta(x,y,z)\,dV$$

And the center of mass is given by

$$\left(\frac{M_x}{\mathrm{Mass}(D)}, \frac{M_y}{\mathrm{Mass}(D)}\right) \qquad \left(\frac{M_x}{\mathrm{Mass}(E)}, \frac{M_y}{\mathrm{Mass}(E)}, \frac{M_z}{\mathrm{Mass}(E)}\right)$$

Ex: Find the center of mass of the lamina D bounded by $y=x^2$, $y=-x^2$, and $x=3$, with density $\delta(x,y) = 2xy^2$

Sol:

The total mass is given by

$$\int_0^3 \int_{-x^2}^{x^2} 2xy^2\,dy\,dx = \int_0^3 \frac{2xy^3}{3}\Big|_{-x^2}^{x^2} dx = \int_0^3 \frac{4x^7}{3} dx = \frac{x^8}{6}\Big|_0^3 =$$

$$\frac{3^7}{2}.$$

The moment about the x-axis M_x is given by

$$\int_0^3 \int_{-x^2}^{x^2} 2x^2y^2\,dy\,dx = \int_0^3 \frac{2x^2y^3}{3}\Big|_{-x^2}^{x^2} dx = \int_0^3 \frac{4x^8}{3} dx = \frac{4}{27}x^9\Big|_0^3 =$$

$$4\cdot 3^6.$$

The moment about the y-axis M_y is given by

$$\int_0^3 \int_{-x^2}^{x^2} 2xy^3\,dy\,dx = \int_0^3 \frac{xy^4}{2}\Big|_{-x^2}^{x^2} dx = 0.$$

Therefore C_m, the center of mass, is

$$\left(\frac{4\cdot 3^6}{\frac{3^7}{2}} = \frac{8}{3},\ 0\right)$$

Ex: Find the center of mass of a sphere of radius 2 with constant density 4

Sol: $\text{Mass} = \iiint_E 4\cdot dV = \frac{64}{3}\pi$

$$M_x = \iiint_E x\,dV = \int_0^{2\pi}\int_0^2\int_0^{\pi} \rho\sin\phi\cos\theta\cdot\rho^2\sin\phi\,d\phi\,d\rho\,d\theta =$$

$$\int_0^{2\pi}\int_0^2 \rho^3\cos\theta\left[\frac{\phi}{2} - \frac{1}{4}\sin(2\phi)\right]\Big|_0^{\pi} d\rho\,d\theta =$$

$$\int_0^{2\pi}\int_0^2 \rho^3\cos\theta\left(\frac{\pi}{2}\right)d\rho\,d\theta = \int_0^{2\pi} \frac{\rho^4}{8}\cdot\pi\cos\theta\Big|_0^2 d\theta =$$

$$\int_0^{2\pi} 2\pi\cos\theta\,d\theta = 2\pi\sin\theta\Big|_0^{2\pi} = 0$$

$M_y = \iiint_E y\,dV = \int_0^{2\pi}\int_0^2\int_0^{\pi} \rho^3 \sin^2\phi \sin\theta \, d\phi\, d\rho\, d\theta =$

$\int_0^2\int_0^{\pi}\int_0^{2\pi} \rho^3 \sin^2\phi \sin\theta \, d\theta\, d\phi\, d\rho = \int_0^2\int_0^{\pi} \rho^3 \sin\phi \, (-\cos\theta)\Big|_0^{2\pi} d\phi\, d\rho$

$= \int_0^2\int_0^{\pi} \rho^3 \sin\theta \, (-1-(-1))\, d\phi\, d\rho = 0$

$M_z = \iiint_E z\,dV = \int_0^{2\pi}\int_0^2\int_0^{\pi} \rho^3 \sin\phi \cos\phi \, d\phi\, d\rho\, d\theta =$

$\int_0^{2\pi}\int_0^2 \rho^3 \, \frac{\sin^2\phi}{2}\Big|_0^{\pi} d\rho\, d\theta = 0.$

Thus the center of mass is $(0,0,0)$.

This is an important observation: If both the shape and the density function are symmetric about $x=0$ / $y=0$ / $z=0$, then $M_x=0$ / $M_y=0$ / $M_z=0$.

We need both: symmetry of the shape is not enough.

Ex: sphere of radius 2, $\delta(x,y,z) = e^{(x^2+y^2+z^2)z}$.

Then $M_x = \iiint z e^{z}\, dV = \int_0^{2\pi}\int_0^2\int_0^{\pi} \rho^4 \sin\phi\, e^{\rho\cos\phi}\, d\phi\, d\rho\, d\theta =$

$\int_0^{2\pi}\int_0^2 -\rho^3 e^{\rho\cos\phi}\Big|_0^{\pi} d\rho\, d\theta$ = (By $u = e^{\rho\cos\phi}$, $du = -\rho\sin\phi\, e^{\rho\cos\phi}\, d\phi$)

$\int_0^{2\pi}\int_0^2 \rho^3 e^{\rho} - \rho^3 e^{-\rho}\, d\rho\, d\theta = 2\pi\int_0^2 \rho^3(e^{\rho} - e^{-\rho})\, d\rho > 0.$