Practice Final Exam

Problem 1: Let $f(x, y) = \frac{6}{1+x^2+y^2}$ for all $x, y \in \mathbb{R}$.
a. Sketch at least 3 level curves of f in the xy-plane.
b. Sketch the graph of f.
c. Find the tangent plane to the graph of f at the point $(1, 1, 2)$.

Problem 2: Show that the ellipsoid $3x^2 + 2y^2 + z^2 = 9$ and the sphere $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ are tangent at the point $(1, 1, 2)$. That is, show that this point is on each of the surfaces and that the tangent planes at $(1, 1, 2)$ of both are equal.

Problem 3: Suppose that $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ are continuously differentiable functions and suppose you know the following facts about f and g:

$$f(1, 2) = (3, 4)$$

$$Df(1, 2) = \begin{pmatrix} 3 & 7 \\ 2 & 6 \end{pmatrix}$$

$$Dg(1, 2) = \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}$$

$$Dg(3, 4) = \begin{pmatrix} 4 & 8 \\ 1 & 2 \end{pmatrix}$$

Find $D(g \circ f)(1, 2)$.

Problem 4: Convert the following sum of integrals in cylindrical coordinates to a single iterated integral in spherical coordinates (you need not evaluate the result):

$$\int_0^{2\pi} \int_0^1 \int_0^r rz \; dzdrd\theta + \int_0^{2\pi} \int_1^{\sqrt{2}} \int_0^{\sqrt{2-r2}} rz \; dzdrd\theta$$

Problem 5:
a. Calculate the work done by the force field $\mathbf{F}(x, y) = (5x^2y, e^y)$ on a particle that moves along the parabola $y = x^2$ from $(-1, 1)$ to $(2, 4)$.
b. Let C be the curve which is the boundary of the region between the parabola $x = y^2 - 4$, the line $x = 0$, and the line $y = 0$. Orient C so that you are traversing it clockwise. Evaluate $\int_C y^2 \; dx + (3xy + \cos(y^3)) \; dy$.

Problem 6: Recall that the volume of a sphere of radius 1 equals $\frac{4}{3}\pi$. Let E be the inside of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Find the volume of E by using a change of variables to compute $\int\int\int_E 1 \; dV$.

Problem 7: Let $\mathbf{F}(x, y, z) = z^2 \; \mathbf{i} + x \; \mathbf{j} + 0 \; \mathbf{k}$, and let S be the part of the parabolic cylinder $z = y^2$ lying between the planes $x = 0$ and $x = 1$ and below the plane $z = 1$. Give S the "outward" pointing normal (its $\mathbf{k}$ component is negative). Evaluate $\int\int_S \mathbf{F} \cdot d\mathbf{S}$.

Problem 8: Evaluate each of the following by using one of Gauss' Theorem or Stokes' Theorem.
a. $\int\int_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = x\ \mathbf{i} + e^{x^2}\mathbf{j} + \mathbf{k}$ and where S is the part of the sphere $x^2 + y^2 + z^2 = 1$ with $z \geq 0$, oriented so that $\mathbf{n}$ is "outward" (away from the origin).
b. $\int_C e^{x^2}\ dx + x\ dy + dz$ where C is given by $\mathbf{x}(t) = (\cos t, \sin t, \cos t + \sin t)$ for $0 \leq t \leq 2\pi$.

Problem 9: Suppose that S is the sphere $x^2 + y^2 + z^2 = a^2$ oriented so that $\mathbf{n}$ is "outward" (away from the origin). Show that

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 0$$

for every C^2 vector field $\mathbf{F}$.