Math 13 Spring 2017
Final Examination
Thursday, June 1st

Name: _____

Instructor: ☐ Prof. Clare ☐ Prof. Song

INSTRUCTIONS

This is a closed book, closed notes exam.

There are 15 problems. You have 3 hours.

Use of calculators is not permitted.

On each open-ended question you must show your work. No credit is given for solutions without supporting calculations. No justification is required for multiple-choice questions.

The Honor Principle requires that you neither give nor receive any aid on this exam.

GOOD LUCK!

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	TOTAL

(1) (*12 points*) For each of the following assertions, select the correct ending.

(a) If $f(x,y)$ is a function, then $\displaystyle\int_0^2 \int_0^x f(x,y)\,dy\,dx = \ldots$

- ☐ $\displaystyle\int_0^2 \int_0^2 f(x,y)\,dx\,dy.$
- ☐ $\displaystyle\int_0^2 \int_0^x f(x,y)\,dx\,dy.$
- ☐ $\displaystyle\int_0^2 \int_0^y f(x,y)\,dx\,dy.$
- ☐ $\displaystyle\int_0^2 \int_y^2 f(x,y)\,dx\,dy.$
- ☐ $\displaystyle\int_0^x \int_0^2 f(x,y)\,dx\,dy.$
- ☐ none of the above.

(b) A potential for the vector field $\mathbf{F}(x,y,z) = \sin(yz^2)\mathbf{i} + xz^2\cos(yz^2)\mathbf{j} + 2xyz\cos(yz^2)\mathbf{k}$ is...

- ☐ $\langle 0\,,\, -xz^4\sin(yz^2)\,,\, -4xy^2z^2\sin(yz^2)\rangle.$
- ☐ $\langle \sin(yz^2)\,,\, xz^2\cos(yz^2)\,,\, 2xyz\cos(yz^2)\rangle.$
- ☐ $x\sin(yz^2) + 2.$
- ☐ $-xz^2(z^2+4y^2)\sin(yz^2).$
- ☐ none of the above.

(c) Let $\mathbf{F}(x, y, z)$ be a vector field defined on an open, connected and simply connected domain. Then $\mathbf{F}$ is conservative if and only if...

☐ $\mathbf{F} = \nabla f$ for some scalar function $f(x, y, z)$.

☐ $\nabla \times \mathbf{F} = \mathbf{0}$.

☐ $\oint_\Gamma \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve Γ in the domain of $\mathbf{F}$.

☐ all of the above.

☐ none of the above.

(d) The Fundamental Theorem of line integrals...

☐ applies to scalar functions.

☐ only applies to conservative vector fields.

☐ applies to all vector fields.

☐ only applies to plane curves, not to curves in 3-dimensional space.

☐ none of the above.

(2) (*7 points*) Calculate

$$\iint_{\mathcal{R}} \frac{x}{1+xy}\, dA$$

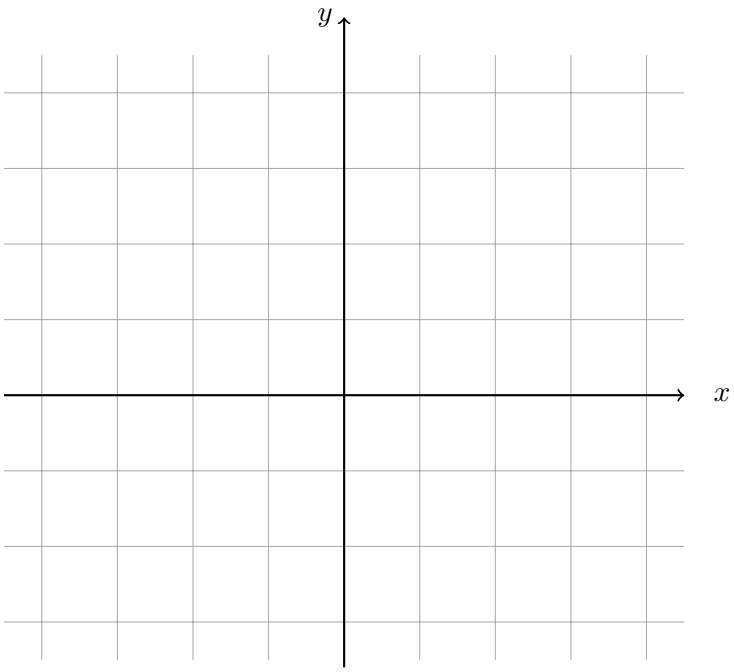
where $\mathcal{R} = [0,1] \times [0,1]$.

Hint: an antiderivative of $\ln x$ *is* $x \ln x - x$.

(3) (*7 points*) Find the volume of the solid bounded by the planes $x = 0$, $y = 2$, $z = 0$, $y = 2x$ and the surface $z = y^2$.

(4) (*3+4=7 points*) Consider the iterated integral $\mathcal{I} = \int_0^3 \int_0^{\sqrt{9-x^2}} (x^2 + y^2)\, dy\, dx$.

(a) Sketch the domain of integration in the xy-plane.



(b) Evaluate $\mathcal{I}$.

(5) (*6 points*) Let $\mathcal{W}$ be a solid with density $\delta(x, y, z)$. Match the following quantities with their expression:

A. Total mass of $\mathcal{W}$

B. y-coordinate of the center of mass

C. Mean value of the density

☐ $\dfrac{\iiint_{\mathcal{W}} x\, dV}{\iiint_{\mathcal{W}} dV}$

☐ $\dfrac{\iiint_{\mathcal{W}} y\, dV}{\iiint_{\mathcal{W}} dV}$

☐ $\dfrac{\iiint_{\mathcal{W}} z\, dV}{\iiint_{\mathcal{W}} dV}$

☐ $\iiint_{\mathcal{W}} 1\, dV$

☐ $\iiint_{\mathcal{W}} \delta(x, y, z)\, dV$

☐ $\dfrac{\iiint_{\mathcal{W}} \delta(x, y, z)\, dV}{\iiint_{\mathcal{W}} dV}$

☐ $\dfrac{\iiint_{\mathcal{W}} x\, \delta(x, y, z)\, dV}{\iiint_{\mathcal{W}} \delta(x, y, z)\, dV}$

☐ $\dfrac{\iiint_{\mathcal{W}} y\, \delta(x, y, z)\, dV}{\iiint_{\mathcal{W}} \delta(x, y, z)\, dV}$

☐ $\dfrac{\iiint_{\mathcal{W}} z\, \delta(x, y, z)\, dV}{\iiint_{\mathcal{W}} \delta(x, y, z)\, dV}$

(6) (*6 points*) Let $\mathcal{D}$ be the domain bounded by the curves $y = x$, $y = 3x$, $xy = 1$ and $xy = 3$ in the first quadrant. What does the integral $\iint_{\mathcal{D}} xy\, dA$ become under the change of variables $x = \dfrac{u}{v}$, $y = v$?

$\square$ $\displaystyle\int_1^3 \int_1^3 u\, dv\, du.$

$\square$ $\displaystyle\int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} u\, dv\, du.$

$\square$ $\displaystyle\int_1^3 \int_{u^2}^{3u^2} u\, dv\, du.$

$\square$ $\displaystyle\int_1^3 \int_1^3 \frac{u}{v}\, dv\, du.$

$\square$ $\displaystyle\int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} \frac{u}{v}\, dv\, du.$

$\square$ $\displaystyle\int_1^3 \int_{u^2}^{3u^2} \frac{u}{v}\, dv\, du.$

(7) (*3+4=7 points*) Let Γ be the curve parametrized by $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + e^t\mathbf{k}$ for $0 \leq t \leq 1$.

(a) Determine the starting point and the end point of Γ.

(b) Consider the function $f(x, y, z) = x\sqrt{y}\ln(1 + x - y + z)$ and let $\mathbf{F} = \nabla f$. Evaluate $\displaystyle\int_\Gamma \mathbf{F} \cdot d\mathbf{r}$.

(8) (*6 points*) Let Σ be the part of the sphere with equation $x^2 + y^2 + z^2 = 1$ in the $x < 0$ region. The integral $\iint_\Sigma x\, dS$ is equal to...

- ☐ $\int_0^\pi \int_0^\pi \cos\theta \sin\varphi \, d\theta \, d\varphi.$
- ☐ $\int_0^\pi \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos\theta \sin\varphi \, d\theta \, d\varphi.$
- ☐ $\int_0^\pi \int_0^{2\pi} \cos\theta \sin\varphi \, d\theta \, d\varphi.$
- ☐ $\int_0^\pi \int_0^{2\pi} \cos\theta \sin^2\varphi \, d\theta \, d\varphi.$
- ☐ $\int_0^\pi \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos\theta \sin^2\varphi \, d\theta \, d\varphi.$
- ☐ $\int_0^\pi \int_0^\pi \cos\theta \sin^2\varphi \, d\theta \, d\varphi.$
- ☐ $\int_0^\pi \int_0^{2\pi} \int_0^1 \cos\theta \sin^2\varphi \, \rho \, d\rho \, d\theta \, d\varphi.$
- ☐ $\int_0^\pi \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^1 \cos\theta \sin^2\varphi \, \rho \, d\rho \, d\theta \, d\varphi.$
- ☐ $\int_0^\pi \int_0^\pi \int_0^1 \cos\theta \sin^2\varphi \, \rho \, d\rho \, d\theta \, d\varphi.$

(9) (*6 points*) Let Σ be the surface parametrized by

$$S(u, v) = \left(u^2\,,\, v^2\,,\, u + 2v\right)$$

for $0 \leq u \leq 2$ and $0 \leq v \leq 2$.

Find an equation for the plane tangent to Σ at the point $(1, 1, 3)$.

Present your answer in the form $ax + by + cz + d = 0$ *(with* a*,* b*,* c*,* d *to be determined).*

(10) (*7 points*) Evaluate $\displaystyle\int_\Sigma \mathbf{F} \cdot d\mathbf{S}$ where

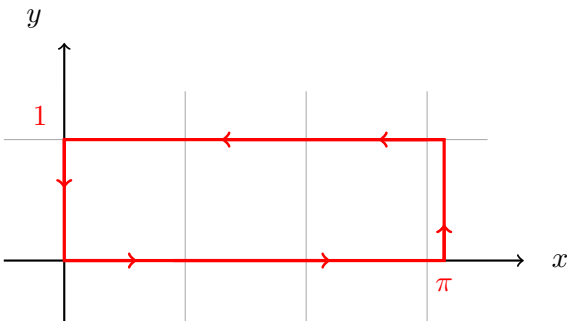
$$\mathbf{F}(x, y, z) = \langle xy\,,\, 4x^2, yz \rangle$$

and Σ is the surface $z = xe^y$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$, oriented in the positive x-direction.

(11) (*7 points*) Calculate the integral of the vector field

$$\mathbf{F}(x,y) = \left\langle y^3 + 2xy + 2y - e^{-x^2}\,,\; x^2 + 3xy^2 + 5x + \cos(\sqrt{y}) \right\rangle$$

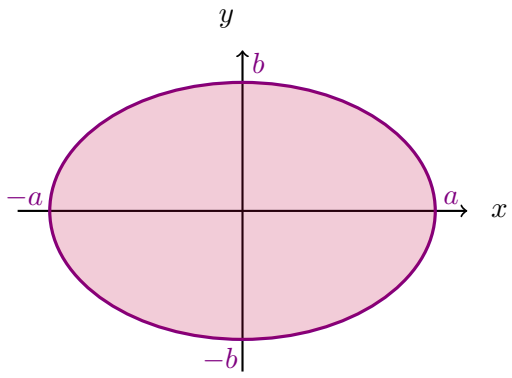
along the closed rectangular path drawn below.



(12) (*7 points*) Use Green's Theorem to calculate the area of the full elliptic domain with equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \leq 1$.

Hint: it might be useful to observe that ***one*** *of the following is a parametrization of the ellipse (for* $0 \leq t \leq 2\pi$*)*

$$\begin{cases} x(t) = a\cos t \\ y(t) = b\sin t \end{cases} \qquad \begin{cases} x(t) = \dfrac{\cos t}{a} \\ y(t) = \dfrac{\sin t}{b} \end{cases} .$$



(13) (*7 points*) Let $\mathbb{S}$ be the hemisphere $x^2 + y^2 + z^2 = 9$ in the $z \geq 0$ region, oriented upward and $\mathbf{F}(x, y, z) = 2x \cos z\mathbf{i} + e^x \sin z\mathbf{j} + xe^y\mathbf{k}$. Evaluate $\iint_{\mathbb{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

(14) (*5 points*) Let Σ denote the part of the paraboloid $y = x^2 + z^2$ with $0 \leq y \leq 1$. Consider

$$\mathbf{F}(x, y, z) = \left\langle z\sqrt{7} + y + e^{x^2}\,,\; xy + \sin\sqrt{y}\,,\; 0 \right\rangle.$$

Evaluate $\displaystyle\iint_{\Sigma} \operatorname{curl}\mathbf{F} \cdot d\mathbf{S}$, using normal vectors pointing in the positive y-direction.

Hint: it might help to use surface invariance.

(15) (*5 points*) Let $\mathbf{F}$ be a vector field defined on a simple solid region $\mathcal{W}$ with boundary the closed surface $\mathcal{S}$. Then, $\iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ is...

☐ a vector.

☐ $\iiint_{\mathcal{W}} \operatorname{div} \mathbf{F}\, dV$.

☐ 0 be the Divergence Theorem.

☐ 0 by Stokes Theorem.

☐ not well-defined.

☐ none of the above.

This page is intentionally left blank. You can use it as scratch paper.

This page is intentionally left blank. You can use it as scratch paper.

This page is intentionally left blank. You can use it as scratch paper.