Homework 8

Elements of solution

Problem 1: **Use Stokes' Theorem to calculate the integral** $\displaystyle\iint_\Sigma \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$ **where**

$$\mathbf{F}(x,y,z) = \langle xz\,,\ yz\,,\ xy\rangle$$

and Σ is the part of the sphere $x^2+y^2+z^2=4$ that lies inside the cylinder $x^2+y^2=1$ and above the xy-plane.
Use the orientation on Σ given by downward-pointing normal vectors.

The boundary of Σ is the intersection of the sphere and the cylinder in the upper half-space: it satisfies the equations $x^2+y^2+z^2=4$ and $x^2+y^2=1$ simultaneously, leading to

$$z = 1 \qquad \text{and} \qquad x^2+y^2=3,$$

which is the circle with center $(0,0,1)$ and radius $\sqrt{3}$ in the $z=1$ plane.
A parametrization for this circle that is compatible with the orientation of Σ by downward-pointing normal vectors is:

$$\begin{cases} x(t) = \sqrt{3}\sin t \\ y(t) = \sqrt{3}\cos t \\ z(t) = 1 \end{cases} \quad , \qquad 0 \le t \le 2\pi \quad .$$

By Stokes' Theorem, we obtain

$$\begin{aligned} \iint_\Sigma \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} &= \int_0^{2\pi} \left\langle \sqrt{3}\sin t\,,\ \sqrt{3}\cos t\,,\ 3\sin t\cos t \right\rangle \cdot \left\langle \sqrt{3}\cos t\,,\ -\sqrt{3}\sin t\,,\ 0 \right\rangle\, dt \\ &= \int_0^{2\pi} (3\sin t\cos t - 3\cos t\sin t + 0)\; dt \\ &= \boxed{0}. \end{aligned}$$

Problem 2: Evaluate the circulation of the vector field

$$\mathbf{F}(x,y,z) = yz\cdot\mathbf{i} + 2xz\cdot\mathbf{j} + e^{xy}\cdot\mathbf{k}$$

along the circle Γ with equation $x^2+y^2=16$ in the $z=5$ plane, oriented counterclockwise as seen from above.

Among the surfaces that admit Γ as their boundary, there is the disk D parametrized in cylindrical coordinates by

$$S(r,\theta) = (r\cos\theta\,,\, r\sin\theta\,,\, 5) \qquad \text{for} \qquad 0\le r\le 4 \quad , \quad 0\le\theta\le 2\pi.$$

Possible normal vectors attached to this parametrization are $\langle 0\,,\,0\,,\,r\rangle$ and $\langle 0\,,\,0\,,\,-r\rangle$. Note that only the first one gives an orientation compatible with that of Γ prescribed by the problem.

The curl of $\mathbf{F}$ is $\langle xe^{xy}-2x\,,\,y-ye^{xy}\,,\,z\rangle$ so Stokes' Theorem gives

$$\begin{aligned}
\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} &= \iint_D \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}\\
&= \int_0^4\int_0^{2\pi} \left\langle r\cos\theta e^{r^2\cos\theta\sin\theta} - 2r\cos\theta\,,\, r\sin\theta - r\sin\theta e^{r^2\cos\theta\sin\theta}\,,\,5\right\rangle\cdot\langle 0\,,\,0\,,\,r\rangle\,d\theta\,dr\\
&= \int_0^4\int_0^{2\pi} 5r\,d\theta\,dr\\
&= \boxed{80\pi}.
\end{aligned}$$

Problem 3: Consider

$$\mathbf{F} = (2x + y\cos\sqrt{z})\cdot\mathbf{i} + (y-\sin xz)\cdot\mathbf{j} + (x+y-z)\cdot\mathbf{k}.$$

Let Σ be the surface consisting of the paraboloid $y=x^2+z^2, 0\le y\le 1$ and the disk $x^2+z^2\le 1, y=1$. Find

$$\iint_\Sigma \mathbf{F}\cdot d\mathbf{S}.$$

Use normal vectors pointing in the negative y-direction on the paraboloid and in the positive y-direction on the disk.

The chosen orientation allows to apply the Divergence Theorem. Since $\operatorname{div}\mathbf{F}=2$, if we call $\mathcal{W}$ the truncated solid paraboloid bounded by Σ, we get

$$\iint_\Sigma \mathbf{F}\cdot d\mathbf{S} = 2\,\mathrm{Vol}(\mathcal{W}) = 2\int_0^{2\pi}\int_0^1\int_{r^2}^1 r\,dy\,dr\,d\theta = 4\pi\int_0^1 r(1-r^2)\,dr = \boxed{\pi}.$$