Math 13 Challenge Problem Solutions

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1. Let $\mathcal{D}$ be the domain bounded by $y = x^2 + 1$ and $y = 2$. Prove the inequality $\frac{4}{3} \leq \iint_{\mathcal{D}} (x^2 + y^2) dA \leq \frac{20}{3}$.

   Solution: Recall that if m is the smallest value of the function f over a domain and M is the largest value, then $m\text{Area}(\mathcal{D}) \leq \iint_{\mathcal{D}} f(x,y)dA \leq M\text{Area}(\mathcal{D})$. To find Area$(\mathcal{D})$, we look at $\int_{-1}^{1}\int_{x^2+1}^{2} dydx = \int_{-1}^{1} 1 - x^2 dx = (x - \frac{1}{3}x^3)|_{-1}^{1} = \frac{4}{3}$. Over this domain, $x^2 + y^2$ is smallest when $x = 0$, $y = 1$, so $m = 1$, and largest when $x = 1$, $y = 2$, so $M = 5$.

2. Verify the Mean Value Theorem for $f(x,y) = e^{x-y}$ on the triangle bounded by $y = 0$, $x = 1$, and $y = x$.

   Solution: We want to find a point $P = (x_0, y_0)$ such that $e^{x_0 - y_0}\text{Area}(\mathcal{D}) = \iint_{\mathcal{D}} e^{x-y} dA$. If we do the integral, we get $\int_0^1 \int_0^x e^{x-y} dydx = \int_0^1 e^x - 1dx = (e-1) - 1 = e - 2$. The area of $\mathcal{D}$ is $\frac{1}{2}$. Then we're trying to find (x_0, y_0) such that $\frac{1}{2}e^{x_0 - y_0} = e - 2$. Multiplying by 2 and taking the natural log, we get $x_0 - y_0 = \ln(2e - 4)$. We just need that (x_0, y_0) is in the triangle. If, for instance, we let $y_0 = .5$, then $x_0 = \ln(2e-4) + .5 \approx .8623$. This point is in our triangle, so we have verified that there exists a point $P \in \mathcal{D}$ such that $e^{x_0 - y_0}\text{Area}(\mathcal{D}) = \iint_{\mathcal{D}} e^{x-y} dA$.

3. Is it true that $\iint_{\mathcal{D}} f(x)g(y)dydx = \left(\int_a^b f(x)dx\right)\left(\int_{h_1(a)}^{h_2(b)} g(y)dy\right)$ for vertically simple regions? Why or why not?

   Solution: We cannot do this. Try for example, working out $\int_1^3 \int_{1/x}^{\sqrt{x}} 2x^2 ydydx = 18$, but $\int_1^3 2x^2 dx \int_1^{\sqrt{3}} ydy = \frac{2}{3}x^3|_1^3 \frac{1}{2}y^2|_1^{\sqrt{3}} = \left(18 - \frac{2}{3}\right)\left(\frac{3}{2} - \frac{1}{2}\right) \neq 18$.

4. Use integrals to calculate the volume of a cone of base radius r and height h.

   Solution: If we let ρ be the radius of any cross-section of the cone, then the volume of the cone is $\int_{z=0}^{h} \pi\rho^2 dz$. Now we just need to express ρ in terms of z. The side of the cone sweeps out in a triangle that follows the line in the xz plane from $(0,h)$ to $(r,0)$, so we have $z = (-h/r)x + h$. Here, if we are along this line, $x = \rho$, so we can solve for ρ to get $\rho = (h-z)(r/h)$. Then our integral becomes $\int_0^h \pi(h-z)^2(r/h)^2 dz = \frac{-r^2\pi}{h^2}\left(\frac{1}{3}(h-z)^3\right)|_0^h = \frac{-r^2\pi}{h^2}\left(0 - \frac{1}{3}h^3\right) = \frac{r^2 h\pi}{3}$.

5. Find the volume of the region contained in the intersection of the cylinders $x^2 + y^2 \leq a^2$ and $x^2 + z^2 \leq a^2$.

   Solution: If we just find the volume in the first octant, then we can multiply that by 8 to get the total volume. In the first octant, the integral becomes $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2}} dzdydx = \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2} dydx = \int_0^a (a^2 - x^2)dx = a^3 - \frac{1}{3}a^3 = \frac{2}{3}a^3$. Then after we multiply this by 8, we get $\frac{16}{3}a^3$.

6. Prove that $\int_0^x \int_0^t F(u)dudt = \int_0^x (x-u)F(u)du$.

   Solution: Note that our variables here are u and t, and x is a constant. If we switch the bounds on the left hand side, then we get $\int_0^x \int_u^x F(u)dtdu$. Since $F(u)$ is a constant with respect to t, we can do the first integral to get $\int_0^x (F(u)t)|_u^x du = \int_0^x (x-u)F(u)du$.

7. Find the volume of the region inside both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.

Solution: If we evaluate this in cylindrical coordinates, since we need to be inside the cylinder, r must go from 0 to 1 and θ from 0 to 2π. Then the integral is $\int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} rdzdrd\theta = \int_0^{2\pi}\int_0^1 2\sqrt{4-r^2}rdrd\theta$. If we do u substitution and let $u = 4 - r^2$, then $du = -2rdr$ and our r bound become 4 to 3, so the integral becomes $\int_0^{2\pi}\int_4^3 u^{1/2}(-1)dud\theta = \int_0^{2\pi} \frac{2}{3}u^{3/2}|3^4 d\theta = \int_0^{2\pi}\frac{2}{3}(8-\sqrt{27})d\theta = \frac{4\pi}{3}(8-\sqrt{27})$.

8. Find the volume of an inverted cone centered at the origin with height H and largest radius R.

   We already solved this problem in one way in problem 4. This time, let's use spherical coordinates. Here, the vertex of the cone is at the origin, and we can call the angle from the z axis to the cone ϕ'. Solving for ϕ' will give us the upper bound for ϕ. Looking at the triangle formed with the z axis, we see that $\tan(\phi') = \frac{R}{H}$, so $\phi' = \tan^{-1}\left(\frac{R}{H}\right)$. We also see that the maximum ρ is the hypotenuse of this triangle, so we get that $\rho = \frac{H}{\cos(\phi)}$. Then our integral becomes $\int_0^{2\pi}\int_0^{\tan^{-1}(R/H)}\int_0^{H/\cos(\phi)} \rho^2\sin(\phi)d\rho d\phi d\theta = 2\pi\int_0^{\tan^{-1}(R/H)} \frac{1}{3}\frac{H^3}{\cos^3(\phi)}\sin(\phi)d\phi = 2\pi\int_0^{\tan^{-1}(R/H)}\frac{1}{3}\frac{H^3}{\cos^2(\phi)}\tan(\phi)d\phi = 2\pi\int_0^{\tan^{-1}(R/H)}\frac{1}{3}H^3\sec^2(\phi)\tan(\phi)d\phi$. If we do u substitution, letting $u = \tan(\phi)$ and $du = \sec^2(\phi)$, then when we adjust our ϕ bounds, we have that when $\phi = 0$, $u = \tan(0) = 0$ and when $\phi = \tan^{-1}(R/H)$, $u = \tan(\tan^{-1}(R/H)) = R/H$. Then our integral becomes $2\pi\int_0^{R/H}\frac{1}{3}H^3 udu = \frac{\pi HR^2}{3}$.