Math 13, Multivariable Calculus

Exercise solution

1. Find the center of mass (x_M, y_M, z_M) of a solid of constant density 1 that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z$, $z = 0$ and $x = 1$

Solution. *The domain can be written as*

$$E = \{(x, y, z) : 1 \le y \le 1, y^2 \le x \le 1, 0 \le z \le x\}$$

The total mass of the solid is given by

$$\begin{aligned}
m &= \int\int\int_E dV = \int_{-1}^{1}\int_{y^2}^{1}\int_0^x dz\,dx\,dy \\
&= \int_{-1}^{1}\int_{y^2}^{1} x\,dx\,dy = \int_{-1}^{1}\left[\frac{x^2}{2}\right]_{x=y^2}^{x=1} dy \\
&= \frac{1}{2}\int_{-1}^{1}(1 - y^4)\,dy = \int_0^1 (1 - y^4)\,dy \\
&= \left[y - \frac{y^5}{5}\right] = \frac{4}{5}
\end{aligned}$$

Since E is symmetricabout the xz- plane, we can immediately say that $y_M = 0$. The other coordinates are:

$$\begin{aligned}
x_M &= \frac{5}{4}\int\int\int_E x dV = \frac{5}{4}\int_{-1}^{1}\int_{y^2}^{1}\int_0^x x dz\,dx\,dy \\
&= \frac{5}{4}\int_{-1}^{1}\int_{y^2}^{1} x^2\,dx\,dy = \frac{5}{4}\int_{-1}^{1}\left[\frac{x^3}{3}\right]_{x=y^2}^{x=1} dy \\
&= \frac{5}{6}\int_0^1 (1 - y^6)\,dy \\
&= \frac{5}{7}
\end{aligned}$$

and

$$\begin{aligned}
z_M &= \frac{5}{4}\int\int\int_E z dV = \frac{5}{4}\int_{-1}^{1}\int_{y^2}^{1}\int_0^x z dz\, dx\, dy \\
&= \frac{5}{4}\int_{-1}^{1}\int_{y^2}^{1}\left[\frac{z^2}{2}\right]_{z=0}^{z=x} dx\, dy = \frac{5}{4}\int_{-1}^{1}\int_{y^2}^{1} x^2\, dx\, dy \\
&= \frac{5}{12}\int_0^1 (1-y^6)\, dy \\
&= \frac{5}{14}
\end{aligned}$$

In the end the coordinates of the center of mass are $(\frac{5}{7}, 0, \frac{5}{14})$.