Math 13, Multivariable Calculus
Practice problems

1. Evaluate the following integral:

$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy.$$

Solution.

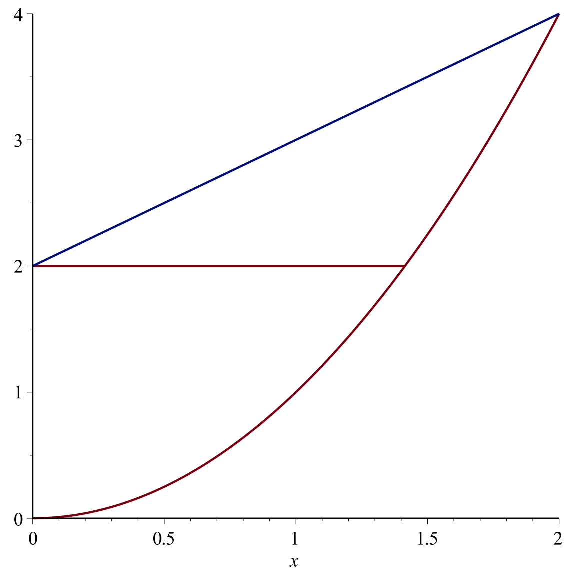
$$\begin{aligned}\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy &= \int_0^2 \int_0^{x^3} e^{x^4}\,dy\,dx \\ &= \int_0^2 e^{x^4} y\Big|_0^{x^3}\,dx = \int_0^2 x^3 e^{x^4}\,dx \\ &= \frac{1}{4}e^{x^4}\Big|_0^2 = \frac{1}{4}(e^{16}-1).\end{aligned}$$

2. In evaluating a double integral over a region D, a sum of iterated integrals was obtained as follows:

$$\iint_D f(x,y)\,dA = \int_0^2 \int_0^{\sqrt{y}} f(x,y)\,dx\,dy + \int_2^4 \int_{y-2}^{\sqrt{y}} f(x,y)\,dx\,dy.$$

Sketch the region D and express the double integral as an iterated integral with reversed order of integration.

Solution. *The first iterated integral corresponds to the integral over the bottom region depicted below, and the second to the upper:*



We reverse the order of integration and rewrite the double integral can be expressed as:

$$\int_0^2 \int_{x^2}^{x+2} f(x,y)\,dy\,dx.$$

3. Evaluate the triple integral $\iiint_T xyz\,dV$, where T is the solid tetrahedron with vertices $(0,0,0), (1,0,0), (1,1,0), (1,0,1)$.

   Solution. *The base of the tetrahedron is a triangle in the xy-plane, and the limits on z are from 0 to the plane which contains the face of the tetrahedron which lies above the base. That plane contains the points* $(0,0,0)$*,* $(1,0,1)$*, and* $(1,1,0)$*, and it is not difficult to determine its equation:* $x-y-z=0$*. Thus*

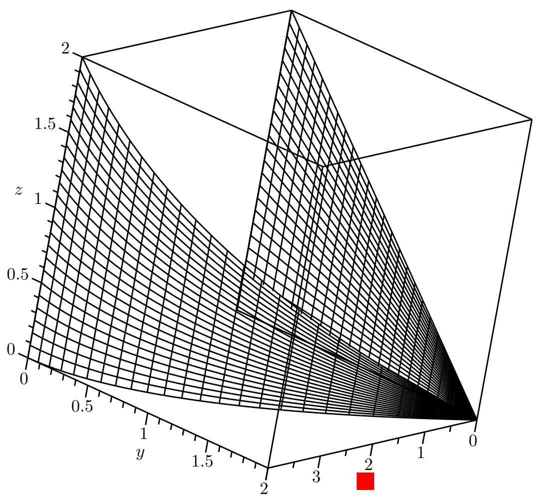
$$\begin{aligned}\iiint_T xyz\,dV &= \int_0^1 \int_0^x \int_0^{x-y} xyz\,dz\,dy\,dx = \int_0^1 x \int_0^x y\frac{(x-y)^2}{2}\,dy\,dx \\ &= \frac{1}{2}\int_0^1 x\left(x^2\frac{y^2}{2} - 2x\frac{y^3}{3} + \frac{y^4}{4}\right)\Big|_{y=0}^{y=x}\,dx = \frac{1}{2}\int_0^1 \left(\frac{x^5}{2} - \frac{2x^5}{3} + \frac{x^5}{4}\right)\,dx \\ &= \frac{1}{2}\frac{1}{12}\frac{x^6}{6}\Big|_0^1 = \frac{1}{144}.\end{aligned}$$

4. Sketch the solid whose volume is given by the following iterated integral, and compute

the value of that volume:

$$\int_0^2 \int_0^{2-y} \int_0^{4-y^2} dx\, dz\, dy.$$

Solution. *View the solid as having a base in the yz-plane which is a triangle determined by the limits:* $0 \le y \le 2$, $0 \le z \le 2-y$, *that is determined by the points* $(0,0,0)$, $(0,2,0)$, *and* $(0,0,2)$. *The solid sits "over" this base and goes from the plane* $x=0$ *to the parabolic cylinder* $x = 4 - y^2$.



The surface is "topped" by the plane $z = 2 - y$, *and completed with pieces of the coordinate planes.*

The volume is

$$\int_0^2 \int_0^{2-y} \int_0^{4-y^2} dx\, dz\, dy = \int_0^2 \int_0^{2-y} (4-y^2) dz\, dy = \int_0^2 (2-y)(4-y^2)\, dy = 20/3.$$

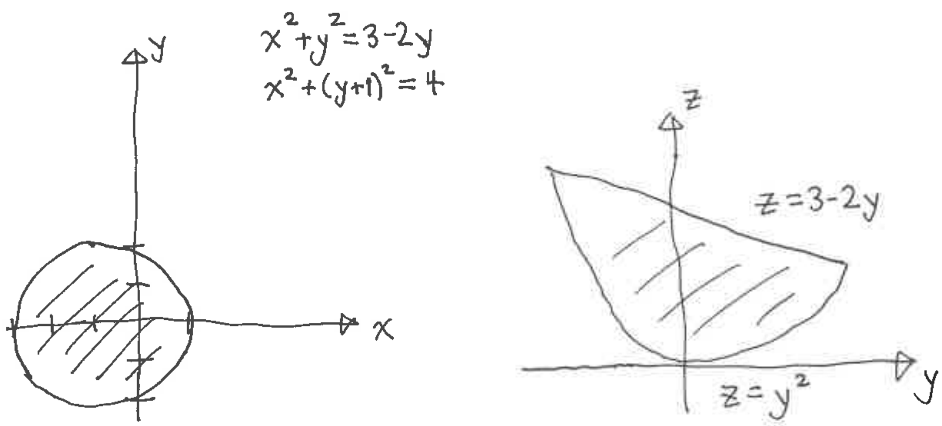
Problem 9. Let E be the three-dimensional region lying below the plane $z = 3 - 2y$ and above the paraboloid $z = x^2 + y^2$.

(a) Sketch the projections onto the xy- and yz-planes.
(b) Sketch a typical cross section parallel to the xz-plane (with y constant).
(c) Sketch the region E.
(d) Set up the limits of integration (but do not integrate!) for the integral

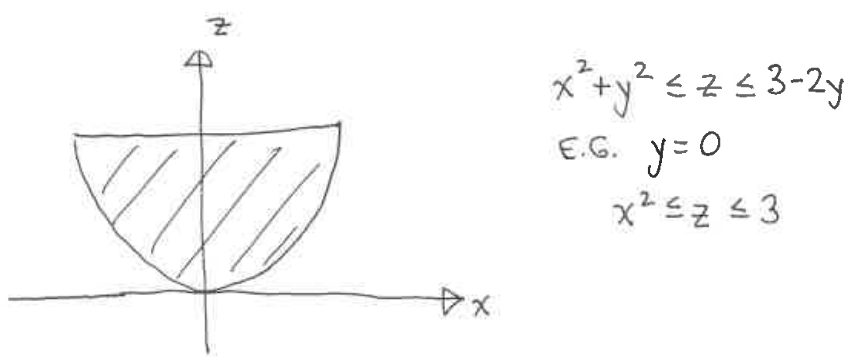
$$\iiint_E f(x, y, z)\, \mathrm{d}V$$

with respect to $\mathrm{d}z\, \mathrm{d}x\, \mathrm{d}y$ and $\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$.

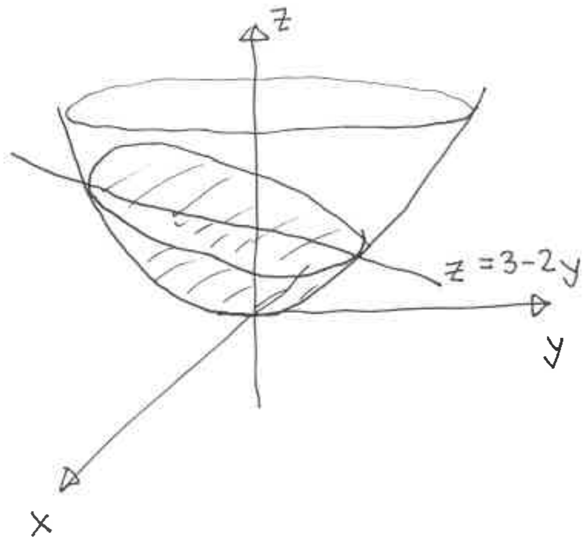
Solution (15.6, Hard). For (a):



For (b):



For (c):



For (d), from the projections and solving $x^2 + (y+1)^2 = 4$ for x to get $y = \pm\sqrt{4-(y+1)^2}$ we have

$$\int_{-3}^{1} \int_{-\sqrt{4-(y+1)^2}}^{\sqrt{4-(y+1)^2}} \int_{x^2+y^2}^{3-2y} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y.$$

On the other hand, from the yz-projection we have to divide up the region into two integrals: the curves $z = y^2$ and $z = 3 - 2y$ intersect at $y^2 + 2y - 3 = (y+3)(y-1) = 0$ so $y = -3, 1$ giving correspondingly $z = 9, 1$. So

$$\int_{0}^{1} \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \int_{1}^{9} \int_{-\sqrt{z}}^{(3-z)/2} \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$