Change of Variables

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Coordinate Transformations in dimension 2

A C^1 function $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that transforms the uv-plane to the xy-plane.

Linear Transformation

A **Linear Transformation** $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$\begin{aligned} T(u,v) &= (au + bv, cu + cv) \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \end{aligned}$$

Here a, b, c, and d are scalar constants.

Linear Transformations map in 2 dimensions parallelograms to parallelograms

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $det(A) \neq 0$. If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$T(u,v) = A \begin{pmatrix} u \\ v \end{pmatrix},$$

then T is one-to-one and onto and it takes parallelograms to parallelograms and the vertices of a parallelogram map to vertices.

If $T(D^*) = D$ then
Area$(D) = |det(A)| \cdot$ (Area(D^*)).

Linear Transformations map in 3 dimensions parallelepipeds to parallelepipeds

In a similar way we can define for every 3×3 matrix A with nonzero determinant. A linear transformation.

The tranformation $T(u, v, w) = A \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ maps parallelepipeds to parallelepipeds.

If $T(D^*) = D$ then
$Volume(D) = |det(A)| \cdot Volume(D^*)$

Important examples of a nonlinear transformation

Polar Coordinates:

$$(x, y) = T(r, \theta) = (r\cos\theta, r\sin\theta)$$

Cylindrical Coordinates:

$$(x, y, z) = T(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$$

The Jacobian of a Transformation in 2D

The **Jacobian** of the tranformation T, denoted

$$\frac{\partial(x,y)}{\partial(u,v)},$$

is the determinant of the derivative matrix $DT(u,v)$.

$$\frac{\partial(x,y)}{\partial(u,v)} = det(DT(u,v)) = det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}.$$

Change of Variables in Double Integrals

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(u,v) = (x(u,v), y(u,v))$ be a coordinate transformation from uv-plane to xy-plane that maps D^* to D. Then

$$\iint_D f(x,y)\,dxdy = \iint_{D^*} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

Double Integrals in Polar Coordinates

$$\iint_D f(x,y)\,dxdy = \iint_{D^*} f(r\cos\theta, r\sin\theta)\mathbf{r}\,drd\theta$$

Note: Jacobian of $T(r,\theta) = (r\cos\theta, r\sin\theta)$ is just r.

$dA = dxdy$ in Cartesian coordinates
$dA = \mathbf{r}drd\theta$ in polar coordinates.

Jacobian in 3D

Coordinate Transformation:

$T(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w))$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}.$$

Change of Variables in Triple Integrals

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$,
$T(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w))$ be a coordinate transformation from uvw-space to xyz-space that maps W^* to W. Then

$$\iiint_W f(x, y)\, dxdydz$$

$$= \iiint_{W^*} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| dudvdw$$

Triple Integrals in Cylindrical Coordinates

$$\iiint_W f(x, y, z)\, dxdydz$$

$$= \iiint_{W^*} f(r\cos\theta, r\sin\theta, z)\mathbf{r}\, drd\theta dz$$

Note: Jacobian of $T(r, \theta) = (r\cos\theta, r\sin\theta, z)$ is just r.

$dA = dxdydz$ in Cartesian coordinates
$dA = \mathbf{r} drd\theta dz$ in cylindrical coordinates.

Triple Integrals in Spherical Coordinates

$$\iiint_W f(x, y, z)\, dxdydz$$

$$= \iiint_{W^*} f(x(\rho, \phi, \theta), y(\rho, \phi, \theta), z(\rho, \phi, \theta))\rho^{\mathbf{2}} \sin \phi\, d\rho d\phi d\theta$$

Note: Jacobian of $T(r, \theta) = (r \cos \theta, r \sin \theta, z)$ is $\rho^{\mathbf{2}} \sin \phi$.

$dA = dxdydz$ in Cartesian coordinates
$dA = \rho^{\mathbf{2}} \sin \phi\, d\rho d\phi d\theta$ in spherical coordinates.