# Chain Rule

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# The chain rule in one variable

Suppose that $x$ is differentiable at $t_0$ and $f$ is differentiable at $x_0$, then the composite function $f \circ x$ is differentiable at $t_0$ and, moreover,

$$(f \circ x)'(t_0) = f'(x_0)x'(t_0)$$

or

$$\frac{df}{dt}(t_0) = \frac{df}{dx}(x_0)\frac{dx}{dt}(t_0)$$

## The chain rule in two variables

Let $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $\mathbf{x}_0 = (x_0, y_0)$ and $\mathbf{x} : T \subseteq \mathbb{R} \to \mathbb{R}^2$ is differentiable at $t_0$. Then

$$\frac{df}{dt}(t_0) = \frac{\partial f}{\partial x}(\mathbf{x}_0)\frac{dx}{dt}(t_0) + \frac{\partial f}{\partial y}(\mathbf{x}_0)\frac{dy}{dt}(t_0)$$

can be rewritten:

$$\frac{df}{dt}(t_0) = \left(\frac{\partial f}{\partial x}(\mathbf{x}_0), \frac{\partial f}{\partial y}(\mathbf{x}_0)\right) \cdot \left(\frac{dx_1}{dt}(t_0), \frac{dx_2}{dt}(t_0)\right)$$

## Generalization to functions $f : X \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$

Let $\mathbf{x} : T \subseteq \mathbb{R} \longrightarrow \mathbb{R}^n$

$$\begin{aligned} \frac{df}{dt}(t_0) &= Df(\mathbf{x}_0)D\mathbf{x}(t_0) \\ &= \nabla f(\mathbf{x}_0) \cdot \mathbf{x}'(t_0) \end{aligned}$$

## Generalization when $\mathbf{x}$ is a surface

$f : X \subseteq \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{x} : T \subseteq \mathbb{R}^2 \to \mathbb{R}^3$

$$\begin{aligned} \frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s} \\ \frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} \end{aligned}$$

## The general chain rule

$\mathbf{f} : X \subseteq \mathbb{R}^m \to \mathbb{R}^p$ and $\mathbf{x} : T \subseteq \mathbb{R}^n \to \mathbb{R}^m$.

$$D(\mathbf{f} \circ \mathbf{x})(\mathbf{t}_0) = D\mathbf{f}(\mathbf{x}_0) D\mathbf{x}(\mathbf{t}_0)$$