The Gradient and Directional Derivatives

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The gradient

Let $f : X \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a scalar valued function. Then the gradient

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right).$$

Directional Derivative

Consider a scalar-valued function f, a point $\mathbf{a}$ in the domain of f and $\mathbf{v}$ any **unit** vector then the **directional derivative of f in the direction of $\mathbf{v}$**, denoted $D_{\mathbf{v}}f(\mathbf{a})$, is

$$D_{\mathbf{v}}f(\mathbf{a}) = \lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{v}) - f(\mathbf{a})}{h}$$

provided the limit exists.

Computing the directional derivative using the gradient

Let f be a differentiable function and $\mathbf{a}$ be a point in the domain of f then

$$\boxed{D_{\mathbf{v}}f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{v},}$$

where $\mathbf{v}$ is a unit vector.

Maximum and minimum values of $D_{\mathbf{v}}f(\mathbf{a})$

- $D_{\mathbf{v}}f(\mathbf{a})$ is maximized when $\mathbf{v}$ points in the **same direction** of the gradient, $\nabla f(\mathbf{a})$.

- $D_{\mathbf{v}}f(\mathbf{a})$ is minimized when $\mathbf{v}$ points in the **opposite direction** of the gradient, $-\nabla f(\mathbf{a})$.

- Furthermore, the maximum and minimum values of $D_{\mathbf{v}}f(\mathbf{a})$ are $\|\nabla f(\mathbf{a})\|$ and $-\|\nabla f(\mathbf{a})\|$, respectively.

Tangent planes to level surfaces: $f(\mathbf{x}) = c$

Let c be any constant.

If $\mathbf{x}_0$ is a point on the level surface $f(\mathbf{x}) = c$, then the vector $\nabla f(\mathbf{x}_0)$ is perpendicular to the surface at $\mathbf{x}_0$.

Computing Tangent plane for level surfaces

Given the equation of a level surface $f(x, y, z) = c$ and a point $\mathbf{x}_0$, then the equation of the tangent plane is

$$\boxed{\nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) = 0}$$

or if $\mathbf{x}_0 = (x_0, y_0, z_0)$ then

$$f_x(\mathbf{x}_0)(x - x_0) + f_y(\mathbf{x}_0)(y - y_0) + f_z(\mathbf{x}_0)(z - z_0) = 0.$$