Vector Fields, Divergence and Curl

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Vector Fields

Definition: A **vector field** on $\mathbb{R}^n$ is a mapping

$$\mathbf{F} : X \subseteq \mathbb{R}^n \to \mathbb{R}^n.$$

Most important vector field: Gradient field

The most important example of a vector field is the gradient of a scalar valued function, $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$

$$\nabla f(x_1, \ldots, x_n) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right).$$

f is called **potential function**.

Question: Given a vector field $\mathbf{F}$ is it possible to find an f such that

$$F = \nabla f?$$

Conservative Vector Field

Let $\mathbf{F}$ be a vector field. Then $\mathbf{F}$ is called conservative if there is a differentiable function f such that

$$\nabla f = \mathbf{F}$$

f is called the potential function for $\mathbf{F}$.

Flow Lines

A **flow line** of a vector field $\mathbf{F}$ is a differentiable path $\mathbf{x}$ such that

$$\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t)).$$

The Del Operator

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

In general

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right).$$

The Divergence

Let $\mathbf{F} : X \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ then the **divergence** is

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \left(\frac{\partial F_1}{\partial x_1}, \frac{\partial F_2}{\partial x_2}, \ldots, \frac{\partial F_n}{\partial x_n} \right),$$

where F_i's are the component functions of the vector field $\mathbf{F}$.

The Curl

Let $\mathbf{F} : X \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ then the **curl** of $\mathbf{F}$ is

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix}.$$

Incompressible vector fields

A vector field $\mathbf{F}$ is called **incompressible** if div $\mathbf{F} = 0$.

If $\mathbf{F} : X \subset \mathbb{R}^3 \to \mathbb{R}^3$ is a differentiable vector field. Then

$$\text{div } (\text{curl } \mathbf{F}) = 0.$$

This says that curl $\mathbf{F}$ is an incompressible vector field.

Irrotational vector fields

A vector field $\mathbf{F}$ in $\mathbb{R}^3$ is called **irrotational** if curl $\mathbf{F} = \mathbf{0}$.

If $f : X \subseteq \mathbb{R}^3 \to \mathbb{R}$ is differentiable, then

$$\text{curl } (\nabla f) = \mathbf{0}.$$

This says that the gradient of f is irrotational.