Derivatives

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Partial Derivative

Partial derivatives are defined as derivatives of a function of multiple variables when all but the variable of interest are held fixed (constant) during the differentiation. Let $f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ then the **partial derivative with respect to** x_i is:

$$\frac{\partial f}{\partial x_i} = \lim_{h \rightarrow 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}$$

we also use f_{x_i} for partial derivative.

Tangent Planes

Let $f : X \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$. If the graph of $z = f(x, y)$ has a tangent plane at $(a, b, f(a, b))$, then the tangent plane has equation

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

REMARK: The existence of a tangent plane to the graph of $z = f(x, y)$ is a stronger condition than the existence of partial derivatives.

$$f(x, y) = ||x| - |y|| - |x| - |y|$$

is a function with partial derivatives at (0,0), but no tangent plane at (0,0). [See the graph]

Good Linear Approximation

We say that

$$h(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is a **good linear approximation** to the function $f : X \subset \mathbb{R}^2 \to \mathbb{R}$ at the point (a, b) if

$$\lim_{(x,y)\to(a,b)} \frac{f(x, y) - h(x, y)}{\|(x, y) - (a, b)\|} = 0$$

Differentiable

A function $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ is **differentiable at** $(a, b) \in X$ if

(1) the partials f_x and f_y exist at (a, b).
and
(2) $h(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$ is a good linear approximation of $f(x, y)$ near (a, b).
A function that is differentiable at all points in the domain is called **differentiable**.

NOTE: We require that X be an open set.

Generalization to $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$

$$h(\mathbf{x}) = f(\mathbf{a}) + f_{x_1}(\mathbf{a})(x_1 - a_1) + \cdots + f_{x_n}(\mathbf{a})(x_n - a_n)$$

is the generalization to the tangent plane. We say that $h(\mathbf{x})$ is a good linear approximation to $f(x, y)$ near $\mathbf{a}$ if

$$\lim_{\mathbf{x} \to \mathbf{a}} \frac{f(\mathbf{x}) - h(\mathbf{x})}{\|\mathbf{x} - \mathbf{a}\|} = 0$$

Differentiability of $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$

We say that $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is **differentiable** at $\mathbf{a}$ if

(1) all partial derivatives f_{x_i} exist at $\mathbf{a}$; and

(2) $h(\mathbf{x})$ is a good linear approximation to $f(\mathbf{x})$ near $\mathbf{a}$.

We say that f is **differentiable** if f is differentiable at every point in the domain X (open set).

The Gradient of $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$

The **gradient** of f is

$$\nabla f(\mathbf{x}) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right)$$

$$h(\mathbf{x}) = f(\mathbf{a}) + f_{x_1}(\mathbf{a})(x_1 - a_1) + \cdots + f_{x_n}(\mathbf{a})(x_n - a_n)$$

can be rewritten

$$\boxed{h(\mathbf{x}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})}$$

Here we think of $\nabla f(\mathbf{a})$ and $\mathbf{x} - \mathbf{a}$ as vectors.

Derivative Matrix for scalar valued functions

Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$, then

$$Df(\mathbf{a}) = [f_{x_1}(\mathbf{a}) \quad f_{x_2}(\mathbf{a}) \quad \cdots \quad f_{x_n}(\mathbf{a})]$$

This is a $1 \times n$ matrix.

We can rewrite,

$$\nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) = Df(\mathbf{a}) \begin{pmatrix} x_1 - a_1 \\ x_2 - a_2 \\ \vdots \\ x_n - a_n \end{pmatrix}$$

General Derivative Matrix

Let $f : X \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function $f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x}))$

$$Df(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

Grand Definition of Differentiability

Let $f : X \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and let $\mathbf{a}$ in X. f is **differentiable at $\mathbf{a}$** if

(1) $Df(\mathbf{a})$ exists and

(2) $\mathbf{h}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) + D\mathbf{f}(\mathbf{a})(\mathbf{x} - \mathbf{a})$ is a good linear approximation to $\mathbf{f}$ near $\mathbf{a}$.

Properties of the derivative

Let f and g be two differentiable functions then

(1) $D(\mathbf{f}+\mathbf{g})(\mathbf{a}) = D(\mathbf{f})(\mathbf{a}) + D(\mathbf{g})(\mathbf{a})$

(2) $D(c\mathbf{f})(\mathbf{a}) = cD\mathbf{f}(\mathbf{a})$ for any scalar c.

If f and g are scalar valued functions:

(1) $D(fg)(\mathbf{a}) = g(\mathbf{a})Df(\mathbf{a}) + f(\mathbf{a})Dg(\mathbf{a})$.

(2) $D(f/g)(\mathbf{a}) = \frac{g(\mathbf{a})Df(\mathbf{a}) - f(\mathbf{a})Dg(\mathbf{a})}{g(\mathbf{a})^2}$.