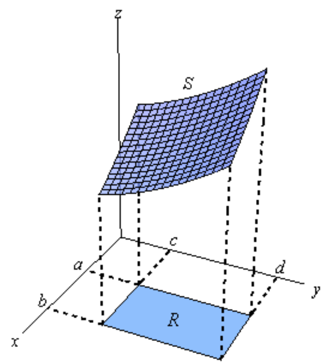
Iterated Integrals and Double Integrals

January 23, 2009

Double Integration



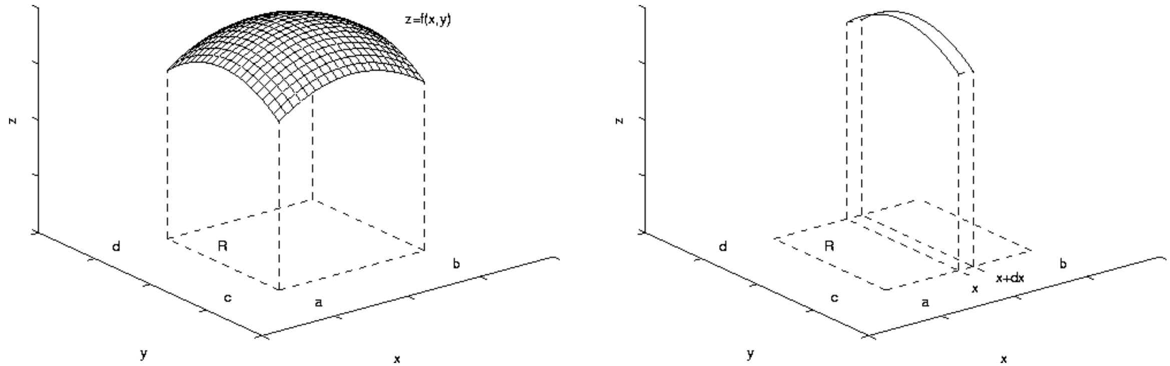
S is described by a function $f(x, y)$ in two variables.
Question: What is the volume under S and over R?

Cavalieri's Principle – The Slicing Method

Let B be a solid and P_x a family of parallel planes such that
(1) B lies between P_a and P_b;
(2) the area of the slice of B cut by P_x is $A(x)$.
Then the volume of B is equal to

$$\int_a^b A(x)dx.$$

Cavalieri's Principle – The Slicing Method



Iterated Integrals

If f is a continuous function and non-negative on a rectangle R,

$$\begin{aligned}\iint_R f(x,y)dA &= \int_a^b\left[\int_c^d f(x,y)\,dy\right]dx \\ &= \int_c^d\left[\int_a^b f(x,y)\,dx\right]dy\end{aligned}$$

The Double Integral

This is an operation that assigns to a function $f(x,y)$ defined and continuous over a region D in the plane a number

$$\iint_D f(x,y)\,dxdy$$

NOTE: If $f(x,y) \geq 0$ for all (x,y) in D, then we can think of this number as the **volume under the graph of f**.

Rectangles

The notation used for rectangles is

$$R = \{(x, y) \in \mathbb{R}^2 | \, a \leq x \leq b, c \leq y \leq d\}$$

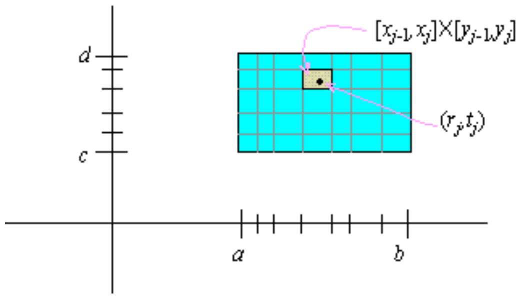
or

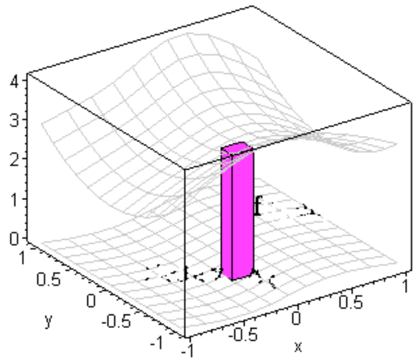
$R = [a, b] \times [c, d]$ - Cartesian Product

Partition of a rectangle

Suppose $R = [a,b] \times [c,d]$, a **partition of R** is a subdivision of R into smaller rectangles. You divide $[a,b]$ into n equally spaced points $a = x_1 < x_2 < \ldots < x_n = b$ and $[c,d]$ into n equally spaced points $c = y_1 < y_2 < \ldots, y_n = d$ and

$$x_{j+1} - x_j = \frac{b-a}{n} \qquad y_{k+1} - y_k = \frac{d-c}{n}.$$


$[x_{j-1}, x_j] \times [y_{j-1}, y_j]$
(r_j, t_j)
d
c
a
b


4
3
2
1
0
1
0.5
0
-0.5
-1
y
-1
-0.5
0
0.5
1
x

Definition of Double Integral over a rectangle

The **double integral** is defined by

$$\iint_R f\, dA = \lim_{\Delta x_i, \Delta y_j \to 0} \sum_{i,j=1}^{n} f(\mathbf{c_{ij}}) \Delta x_i \Delta y_j$$

provided the limit exists. If the limit exists we say that f is **integrable** on R.

Integrability

- If f is continuous on the closed interval R, then $\iint_R f(x,y)\,dA$ exists.

- If f is bounded on R and the set of discontinuities of f has zero area, then $\iint_R f(x,y)\,dA$ exists.

NOTE: Here $dA = dxdy$ or $dA = dydx$.

Fubini's Theorem

Let f be integrable on a rectangle

$$R = [a, b] \times [c, d],$$

then $\iint_R f(x, y)\, dA$ can be computed using the method of iterated integrals.

Properties of the Double Integral

- If $f+g$ is integrable, then $\iint_R (f+g)\,dA = \iint_R f\,dA + \iint_R g\,dA$;

- If c is a scalar, then $\iint_R c\,f\,dA = c\iint_R f\,dA$

- If $f(x,y) \le g(x,y)$ in R, then $\iint_R f(x,y)\,dA \le \iint_R g(x,y)\,dA$;

- If $|f|$ is integrable on R then $|\iint_R f(x,y)\,dA| \le \iint_R |f(x,y)|\,dA$.