WRITTEN HOMEWORK #1, DUE 4/2/2012 AT 4PM

You can turn this in during class on April 2 or at my office (Kemeny 316) by 4pm. Please make sure your homework assignment is stapled, if necessary, before handing it in. (In particular, there is no guarantee that a stapler will be in our classroom.) Do not use paper clips or the technique where you rip up a small piece of the upper left hand corner.

Solutions should be justified in a reasonably rigorous way; if you are unsure how much detail you need to provide, you can ask me before turning in your assignment and I can give you some indication of whether you are on the right track.

Since I discovered that solutions to many of the problems in Stein and Shakarchi are available online, a lot of our problems will come from different, unspecified sources. Please do not try to search for solutions on the Internet; if a problem from Stein and Shakarchi is assigned (which we will do occasionally) definitely do not consult solutions on the Internet at any point in time.

(1) For each of the following equations, give a geometric description of the set of complex numbers (ie, describe how this set looks in the complex plane) which solve that equation. The numbers $z_1, z_2, \ldots$ refer to arbitrary, distinct, fixed complex numbers.
   (a) $|z - z_1| = |z - z_2|$. Your description should involve z_1, z_2.
   (b) $\frac{c}{z} = \overline{z}$, where $c > 0$ is a fixed real number.
   (c) $|z - z_1| + |z - z_2| = c$, where $c > 0$ is a fixed real number. (Hint: the shape depends on whether $c < |z_1 - z_2|, c = |z_1 - z_2|$, or $c > |z_1 - z_2|$.)
   (d) $|z| = \operatorname{Im} z + 1$.

(2) For each of the following equations in z, find all complex solutions. You may leave your answers in either rectangular or polar form.
   (a) $z^2 + iz - 2 = 0$.
   (b) $z^4 = -1$.
   (c) $z^3 = -\sqrt{3} - i$.
   (d) $e^z = e^2$. (Hint: the answer is not just $z = 2$.)

(3) (a) Recall that an nth root of unity is any solution to $z^n = 1$. If ζ_n is an nth root of unity not equal to 1, show that $1 + \zeta_n + \zeta_n^2 + \ldots + \zeta_n^{n-1} = 0$.
   (b) Let P be a regular n-gon inscribed in the unit circle. Fix one of the vertices, and consider the $n-1$ diagonals obtained by connecting that vertex to the remaining $n - 1$ vertices. Show that the product of the lengths of these diagonals is equal to n. This is a rather remarkable geometric fact which is perhaps most easily proved using complex numbers! (Hint: there is a way to use the previous part of this problem...)

(4) Let z_1, z_2, z_3 be distinct complex numbers. Show that z_1, z_2, z_3 are the vertices of an equilateral triangle in the complex plane if and only if $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_1z_3 + z_2z_3$. (Hint: one approach is to solve the problem in the special case where $z_1 = 0$, and then use this to solve the general case.)

If you are interested in improving your skills, Stein and Shakarchi exercises 2, 3, 4, 5, 6, and 7 are all worth doing. Exercises 5 and 6 are essentially topology problems. Exercise 1 is similar to the first problem on this assignment.