LINE INTEGRALS OF VECTOR FUNCTIONS: GREEN'S THEOREM

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1. A differential criterion for conservative vector fields

We would like to be able to determine whether $\mathbf{F}$ is conservative without too much difficulty. However, the path-independence property for conservative fields does not help at all with this problem, since in practice it is impossible to check that an integral is independent of path for EVERY choice of starting and end point and EVERY choice of path connecting these two points. In one example we saw how we could try to calculate 'partial integrals' to either find a potential function, or rule out its existence. However, this requires calculating integrals, which in general can be a fairly difficult problem.

In an earlier example, we showed that a field was not conservative by assuming that it was, and then showing that this led to a contradiction. More specifically, suppose $\mathbf{F} = \langle P, Q\rangle$ is conservative, so that $\mathbf{F} = \nabla f$, and make the additional assumption that $\mathbf{F}$ is C^1; ie, P, Q have continuous first-order partial derivatives. Then $f_x = P, f_y = Q$, and we can apply Clairaut's Theorem to conclude that $f_{xy} = P_y = f_{yx} = Q_x$. In other words, if $\mathbf{F} = \langle P, Q\rangle$ is conservative and C^1, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Any conservative vector field satisfies the above property, which only involves taking derivatives, not integrals. As such, this looks like it is a better test for whether a vector field is conservative or not than anything else we know. However, there is one major problem: not every field which passes this test is conservative! In other words, if $P_y \neq Q_x$ for even one point in D, then we know that $\mathbf{F}$ is not conservative on D, but even if $P_y = Q_x$ everywhere on D, we cannot necessarily conclude that $\mathbf{F}$ is conservative.

Example. (This is Problem #33 from Chapter 17.3, but the example is so classical that it appears in many sources.) Let $\mathbf{F}(x, y)$ be defined by

$$\mathbf{F} = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}.$$

This vector field is defined on every point of $\mathbb{R}^2$ except the origin. Its coordinate functions are defined by

$$P(x,y) = \frac{-y}{x^2+y^2}, Q(x,y) = \frac{x}{x^2+y^2}.$$

If we calculate P_y, Q_x, we find they are equal:

$$P_y = \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}, Q_x = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

On the other hand, if we let C be the path given by $\mathbf{r}(t) = \langle \cos t, \sin t\rangle, 0 \leq t \leq 2\pi$ – namely, the closed path given by the unit circle in the counterclockwise direction, then the integral of $\mathbf{F}$ along C is equal to

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle -\sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle\, dt = \int_0^{2\pi} 1\, dt = 2\pi \neq 0.$$

Therefore, this vector field is not path-independent, and hence is not conservative on D, even though $P_y = Q_x$.

However, it turns out there is still a way to partially salvage this criterion for being a conservative vector field. A closed curve C is called a *simple closed curve* if it does not intersect itself anywhere; in terms of a parameterization $\mathbf{r}(t), a \leq t \leq b$, this means that $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ for any $t_1 \neq t_2$ except at $t_1 = a, t_2 = b$ or vice versa. A simple closed curve splits up $\mathbb{R}^2$ into a region contained in the curve and a region outside the curve; both these regions are connected. (Even though this seems obvious, proving that this is true is not easy and was not done until the 19th century by Camille Jordan!)

A connected set D is called *simply connected* if, given any simple closed curve lying in D, the interior of that curve only contains points in D. Another alternate definition is that a set D is simply connected if, given any closed curve in D, it is possible to continuously shrink the curve D to a point with out ever leaving the set D. In both these definitions, the intuitive idea behind a simply connected set is that it is a set with no holes in it. We have no tools for rigorously showing that a set is simply connected or not, but in practice it is usually easy to 'intuitively see' if a set is simply connected. (If you want to learn how to make your intuition precise, a good place to start is to take a topology class.)

Examples.

- The set $x^2+y^2 < 1$ is simply connected; intuitively it has no holes.
- The set D equal to $\mathbb{R}^2 - (0,0)$; ie, the plane with the origin removed, is not simply connected, because of the hole at the origin. For example, the curve C we looked at in the previous example is a simple closed curve lying entirely in D, but its interior contains a point not in D.
- The annulus $1 \leq x^2+y^2 \leq 4$ is not simply connected. Any circle going around the annulus will contain points which are not in the annulus itself.

In the example above where $P_y = Q_x$, yet $\mathbf{F} = \langle P, Q\rangle$ was not a conservative vector field, we saw that $\mathbf{F}$ was only defined on a set D which was not simply connected. It turns out that if $P_y = Q_x$ is true for all points D on an open, simply-connected region, then $\mathbf{F}$ is conservative!

Theorem. Let $\mathbf{F} = \langle P, Q \rangle$ be a C^1 vector field on an open, simply-connected set D. If $P_y = Q_x$ for all points in D, then $\mathbf{F}$ is conservative.

2. Green's Theorem

We now discuss a theorem which connects double integrals with line integrals. Recall that a simple closed curveis a closed curve which does not intersect itself. Let C be a simple closed curve lying in $\mathbb{R}^2$. Then the *positive orientation* of C is defined to be the orientation of C we obtain by moving in single counterclockwise loop along C. An alternate definition is that if we walk in the direction of the positive orientation for C, the interior of C is always on our left-hand side. The *negative orientation* of C is the opposite orientation of the positive orientation for C.

Theorem. (Green's Theorem) Let C be a positively oriented, piecewise smooth, simple closed curve in $\mathbb{R}^2$. Let D be the region bounded by C. If $\mathbf{F} = \langle P, Q \rangle$ is a C^1 vector field on D, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

An alternate formulation of Green's Theorem is

$$\int_C P\,dx + Q\,dy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

- We will not prove Green's Theorem, or give an indication of how the proof goes yet, but we will see how it is a consequence of a more general theorem we will study in a few weeks.
- Green's Theorem should be philosophically interpreted as a higher-dimensional analogue of the FTC. The usual FTC relates the value of a double integral over an interval to the value of its antiderivative at the endpoints, which is also the boundary, of that interval. Green's Theorem relates the value of the double integral of the expression $Q_x - P_y$ over a region D to a line integral of a related function (which may not look like an antiderivative, but certainly involves 'partial integrals' of Q_x, P_y) over the boundary of D.
- Green's Theorem can sometimes act as a bridge between line integrals and double integrals. For example, it may be difficult or tedious to evaluate a certain line integral, but an application of Green's Theorem might convert that line integral into a double integral which is easier to evaluate. Conversely, a double integral which might look difficult to evaluate can sometimes be converted to a line integral which is easier to evaluate, although this is slightly more difficult to do.
- The strategy of when to use Green's Theorem: In general, if you have to evaluate a line integral over a rectangle, or the boundary of some other simple two-dimensional region which consists of several different pieces, using Green's Theorem will usually simplify your calculation. Calculating a line integral over

a rectangle involves breaking up that rectangle into its four sides, separately calculating parameterizations for each side, and then calculating four different line integrals. However, an application of Green's Theorem will convert the line integral into a single double integral over a rectangle, which usually is easy to do.

Line integrals over regions like circles may or may not be simplified using Green's Theorem. It depends on the vector field being integrated. If you find you are having difficulty evaluating a certain line integral because the resulting integrand is excessively complicated, try using Green's Theorem to see if you get a simple double integral.

Examples.

- Let $\mathbf{F} = \langle y\cos x, x^2\rangle$, and let C be the boundary of the square R, given by $0 \le x \le 1, 0 \le y \le 1$ with positive orientation. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$.

  If you wanted to, you could split C up into its four sides, parameterize each side, and then evaluate the line integral along each side, but that's a lot of work. If you apply Green's Theorem, with $P = y\cos x, P_y = \cos x, Q = x^2, Q_x = 2x$, then we get

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_R Q_x - P_y\, dA = \int_0^1\int_0^1 2x - \cos x\, dy\, dx = \int_0^1 2x - \cos x\, dx = x^2 - \sin x\Big|_0^1 = 1 - \sin 1.$$

- Let $\mathbf{F} = \langle -y^3 + \log(2+\sin x), x^3 + \arctan y\rangle$, and let C be the circle $x^2+y^2 = 1$ with counterclockwise orientation. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$.

  If you try to directly calculate this line integral, you will have a really difficult time because the resulting integral in the parameter t will be a complete mess. If you try to evaluate this integral using the fundamental theorem for line integrals you will also fail, because $\mathbf{F}$ is not conservative. Therefore, you should try Green's Theorem.

  Since $P = -y^3 + \log(2+\sin x), P_y = -3y^2$. Similarly, $Q_x = 3x^2$, so Green's Theorem says

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_D Q_x - P_y\, dA = \iint_D 3x^2 + 3y^2\, dA,$$

  where D is the disc $x^2+y^2 \le 1$. This looks like an integral we should evaluate using polar coordinates. If we convert this double integral to polar coordinates we get

$$\iint_D 3x^2 + 3y^2\, dA = \int_0^{2\pi}\int_0^1 3r^2 r\, dr\, d\theta = \int_0^{2\pi} 3/4\, d\theta = \frac{3\pi}{2}.$$

- An interesting application of Green's Theorem is to the calculation of areas of two-dimensional regions. Recall that the area of a region D can be expressed as the value of the double integral $\iint_D dA$. If we select P, Q such that $Q_x - P_y =$

1, then $\int_C P\,dx + Q\,dy$ will also equal the area of D, where C is the boundary of D with positive orientation.

For example, selecting $Q = x, P = 0$ yields the equation $A(D) = \int_C x\,dy$. Selecting $Q = 0, P = -y$, or $Q = x/2, P = -y/2$ gives the equations

$$A(D) = -\int_C y\,dx = \frac{1}{2}\int_C x\,dy - y\,dx.$$

Let us apply this to the calculation of the area of an ellipse (which we already know how to do using other means). Suppose D is the region $x^2/a^2 + y^2/b^2 \leq 1$; this is the region enclosed by an ellipse with axes of length $2a, 2b$. The boundary C of D is parameterized by $\mathbf{r}(t) = \langle a\cos t, b\sin t\rangle, 0 \leq t \leq 2\pi$. If we use the last of the expressions for the area of D, we get

$$A(D) = \frac{1}{2}\int_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi}(a\cos t)(b\cos t) - (b\sin t)(-a\sin t)\,dt = \frac{1}{2}\int_0^{2\pi} ab\,dt = \pi ab.$$

- (Exercise 17.5.21 of the textbook) This problem from the textbook gives a neat application of Green's theorem to the problem of calculating the volume of a polygon. More specifically, the problem describes a simple formula for the volume of a polygon in terms of the coordinates of the vertices of the polygon. The problem consists of two parts (really three, but the last part is not as interesting): first, show that if C is the line segment from (x_1, y_1) to (x_2, y_2), then

$$\int_C x\,dy - y\,dx = x_1y_2 - x_2y_1,$$

and then if $(x_1, y_1), \ldots, (x_n, y_n)$ are the vertices of a polygon, listed in counterclockwise order, then its area A is equal to

$$A = \frac{1}{2}((x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \ldots + (x_ny_1 - x_1y_n)).$$

We first find a parameterization for C. For example, $\mathbf{r}(t) = \langle x_1(1-t) + x_2(t), y_1(1-t) + y_2t\rangle, 0 \leq t \leq 1$, works. This choice of parameterization has

$$x'(t) = x_2 - x_1, y'(t) = y_2 - y_1,$$

so the line integral in question is equal to

$$\int_0^1 (x_1(1-t) + x_2(t))(y_2 - y_1) - (y_1(1-t) + y_2t)(x_2 - x_1)\,dt$$

If you expand the terms in the integrand, you will find that all the terms with coefficient t cancel out, and there is also some cancellation in the constant terms. The end result is

$$\int_0^1 (x_1y_2 - x_2y_1)\,dt = x_1y_2 - x_2y_1,$$

as desired.

For the second part, recall that if we have a positively oriented simple closed curve C enclosing a region D, the area of D is given by the expression

$$\frac{1}{2}\int_C x\,dy - y\,dx.$$

(This was an application of Green's Theorem where we chose P, Q in a special way to get $Q_x - P_y = 1$.) In this problem, if we let C_i be the segment from (x_i, y_i) to (x_{i+1}, y_{i+1}) (if $i = n$, we let $x_{n+1} = x_1, y_{n+1} = y_1$), then C is the same path as $C_1, C_2, \ldots, C_n$ in succession. Therefore,

$$\int_C x\,dy - y\,dx = \int_{C_1} x\,dy - y\,dx + \int_{C_2} x\,dy - y\,dx + \ldots + \int_{C_n} x\,dy - y\,dx.$$

If we replace each line integral on the right hand side with the corresponding term we get from part (a), and then multiply by $1/2$, we get the desired result.

As an example, consider the triangle with vertices at $(0,0), (4,3), (5,2)$. Then an application of this formula gives an area of $7/2$. This method of calculating the area is easier than using the basic formulas from Euclidean geometry (though possible; give it a try!). Also, if you are attentive, you will notice that in this example, the formula reduces to essentially the formula for the area of a parallelogram in terms of the determinant of a 2×2 matrix.

- Suppose $\mathbf{F} = \langle P, Q \rangle$ is conservative on D. Then $P_y = Q_x$, so an application of Green's Theorem gives

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D Q_x - P_y\,dA = \iint_D 0\,dA = 0.$$

This is exactly as we expect by the FTC for line integrals, so in some sense Green's Theorem is a generalization of the FTC for line integrals, at least for regions D enclosed by simple closed curves.