INTEGRATION OVER NON-RECTANGULAR REGIONS

Contents

1. A slightly more general form of Fubini's Theorem

We now want to learn how to calculate double integrals over regions in the plane which are not necessarily rectangles. Suppose we have a bounded region D in $\mathbb{R}^2$ whose boundary is a piecewise smooth curve; for example, D might be a circle, an ellipse, a polygon, or a region defined by polynomial inequalities. How do we integrate a function $f(x, y)$ over D?

The answer is to define a new function $F(x, y)$, which is equal to $f(x, y)$ on D and 0 everywhere else. If $f(x, y)$ is continuous, then $F(x, y)$ will not necessarily be continuous but it can only be discontinuous at the boundary of D. We can then enclose D with any rectangle R, and then calculate

$$\iint\limits_R F(x, y)\, dA$$

Because $F(x, y) = 0$ outside of D, this double integral still represents the volume of the solid over D and under $z = f(x, y)$. It turns out that Fubini's Theorem still holds true for functions which are continuous everywhere except possibly at a finite number of smooth curves, if both iterated integrals exist. In particular, we can use iterated integrals to calculate integrals over regions in the plane which are called *Type I* and *Type II* regions.

A Type I region is a region in the xy-plane defined by inequalities of the form $a \leq x \leq b$, for some a, b, and $g_1(x) \leq y \leq g_2(x)$, for some continuous functions $g_1(x), g_2(x)$ on $[a, b]$. A Type II region is a region in the xy-plane of the form $c \leq y \leq d, h_1(y) \leq x \leq h_2(y)$ for two continuous functions $h_1(y), h_2(y)$ on $[c, d]$.

Examples.

- A rectangle $R = [a, b] \times [c, d]$ is both a type I and type II region.
- A disc, such as $x^2 + y^2 \leq 1$, is both a type I and type II region. For example, a disc can be described in the form of a type I region with the inequalities $-1 \leq x \leq 1, -\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}$.
- The region between the parabola $y = x^2$ and the interval $[-1, 1]$ on the x-axis is a type I, but not a type II region. It can be described using $-1 \leq x \leq 1, 0 \leq y \leq x^2$, but there is no way of describing this region as a type II region. One can see this intuitively since the intersection of a type II region with a horizontal line will always be zero or one line segment, but the intersection

of this region with the horizontal line segment $y = 1/2$ consists of two line segments.

- The region $1 \leq x^2 + y^2 \leq 4$, which is an annulus (a disc with a hole in the middle), is neither a type I nor a type II region. Nevertheless, we can cut this region up in such a way so that it becomes a (disjoint) union of type I and/or type II regions. For example, splitting this region in half using the line $y = 0$ cuts it apart into two type I regions.

Suppose we have a type I region D, given by $a \leq x \leq b, g_1(x) \leq y \leq g_2(x)$. How do we calculate the integral of a function $f(x, y)$ over this region? The cross-sectional area of the volume with coordinate x is evidently given by

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy$$

so the double integral of $f(x, y)$ over D is given by

$$\iint\limits_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

Notice that the bounds of integration in the inner integral are no longer numbers, but functions of x. Pay close attention to the order of integration – if functions of x appear in the bounds of the inner integral, then we must integrate with respect to y first. In general, we are only allowed to place variables in the bounds of integration of an integral if we have not yet integrated with respect to that variable yet. In a similar way, if we have a type II region D defined by $c \leq y \leq d, h_1(y) \leq x \leq h_2(y)$, then the double integral of $f(x, y)$ over D is given by

$$\iint\limits_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

Examples.

- Calculate the integral of $f(x, y) = x^2 + y^2$ over the region defined by $0 \leq x \leq 1, 0 \leq y \leq x$. This region is a type I region (and also a type II region), with $g_1(x) = 0, g_2(x) = x$. Then the integral in question is equal to

  $$\int_0^1 \int_0^x x^2 + y^2\, dy\, dx.$$

  We calculate the inner integral, with respect to the variable y, and then evaluate the resulting function at the bounds of integration, which are now functions of x:

  $$\int_0^1 \left(yx^2 + \frac{y^3}{3} \Big|_{y=0}^{y=x} \right) dx = \int_0^1 x^3 + \frac{x^3}{3}\, dx = \int_0^1 \frac{4x^3}{3}\, dx.$$

  Notice that after evaluating the inner integral, we are still left with a function of x, but that this function depends on the actual bounds of integration in the inner integral. In this example, we can easily evaluate the second integral we need to calculate:

$$\int_0^1 \frac{4x^3}{3}\,dx = \frac{x^4}{3}\Big|_0^1 = \frac{1}{3}.$$

- Sometimes the region you are integrating over is not explicitly described as a type I or type II region, and you need to determine the description yourself. For example, suppose you want to calculate the integral of $f(x,y) = x+y$ over the region bounded by $y = x$ and $y = x^2$. The first step is to sketch the region of integration, which in this example involves sketching $y = x, y = x^2$. After sketching this region, one quickly sees that this region has a simple description in terms of a type I function: $0 \le x \le 1$ (these are the x-coordinates at which $y = x, x^2$ intersect), $x^2 \le y \le x$ (since $y = x^2$ is the lower boundary of the region and $y = x$ is the upper boundary of the region). We can then setup the iterated integral we want to calculate:

$$\int_0^1 \int_{x^2}^{x} (x+y)\,dy\,dx.$$

Evaluate the inner integral:

$$\int_0^1 \left(xy + \frac{y^2}{2}\Big|_{y=x^2}^{y=x} \right) dx = \int_0^1 \left(x^2 + \frac{x^2}{2} \right) - \left(x^3 + \frac{x^4}{2} \right) dx = \int_0^1 \frac{3x^2}{2} - x^3 - \frac{x^4}{2}\,dx.$$

Now we evaluate the outer integral:

$$\int_0^1 \frac{3x^2}{2} - x^3 - \frac{x^4}{2}\,dx = \frac{x^3}{2} - \frac{x^4}{4} - \frac{x^5}{10}\Big|_0^1 = \frac{1}{2} - \frac{1}{4} - \frac{1}{10} = \frac{3}{20}.$$

- Consider the tetrahedron bounded by $z = 2 - 2x - y, x = 0, y = 0, z = 0$. Write down two iterated integrals, one over a type I region, the other over a type II region, which equal the volume of this tetrahedron. Evaluate both integrals to check that they are equal, and then check that this matches with the answer given by geometry.

  We start by sketching the volume we want to calculate. In this case, we are looking at the volume of a solid which is bounded above by $z = 2 - 2x - y$, below by $z = 0$, and over the region in the xy plane defined by $x \ge 0, y \ge 0, 2x + y \le 2$. As a type I region, this has a description $0 \le x \le 1, 0 \le y \le 2 - 2x$, and as a type II region, this has a description $0 \le y \le 2, 0 \le x \le 1 - y/2$. Therefore, the iterated integral we want to evaluate, using the type I description, is

$$\begin{aligned}\int_0^1 \int_0^{2-2x} (2 - 2x - y)\,dy\,dx &= \int_0^1 \left((2-2x)y - \frac{y^2}{2}\Big|_{y=0}^{y=2-2x} \right) dx \\ &= \int_0^1 \frac{(2-2x)^2}{2}\,dx = \frac{(2-2x)^3}{6} \cdot \frac{1}{-2}\Big|_0^1 = \frac{8}{12} = \frac{2}{3}.\end{aligned}$$

Using the type II description of the region of integration gives the following iterated integral:

$$\begin{aligned}\int_0^2\int_0^{1-y/2} 2-2x-y\,dx\,dy &= \int_0^2 \left(x(2-y) - x^2\Big|_{x=0}^{x=1-y/2} \right) dx \\ &= \int_0^2 \frac{(2-y)^2}{2} - \frac{(2-y)^2}{4}\,dy = \int_0^2 \frac{(2-y)^2}{4}\,dy \\ &= \frac{-(2-y)^3}{12}\Big|_0^2 = \frac{8}{12} = \frac{2}{3}.\end{aligned}$$

As expected (by Fubini's Theorem), we get the same answer regardless of the order in which we do the integration.

To check that this answer accords with geometry, notice that the tetrahedron can be thought of as having base given by the triangle in the xy plane bounded by $x = 0, y = 0, 2x + y = 2$. This is a right-angle triangle with area 1, since it has base of length 1 and height 2. The height of the tetrahedron is 2, since $z(0, 0) = 2$, and the formula for the volume of a tetrahedron is the area of the base times height divided by 3, which in this case comes out to $2(1)/3 = 2/3$, as expected.

In the last example, we saw how we were able to express a double integral over a region which was both type I and type II as two different iterated integrals, where the order of integration is different. The technique where we take an iterated integral and re-express the integral in the other order of integration is sometimes called *interchanging the order of integration*. A useful technique when you are asked to interchange the order of integration is to start by sketching the region you are integrating over, using the bounds of the initial iterated integral as your place to start.

Example. Sketch the region of integration of the following iterated integral, and then interchange the order of integration:

$$\int_0^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\,dy\,dx.$$

Because we are integrating with respect to y first and then x (remember, we determine this by looking at the order of the differentials in the iterated integral), we know that the bounds on the outer integral are with respect to the variable x, while the bounds on the inner integral are with respect to y. That is, the region D we are integrating over is given by the inequalities $0 \le x \le 2, -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$. A sketch of this region reveals that it is the right half of a circle with radius 2, centered at the origin. We interchange the order of integration by rewriting this region as a type II region.

If we think of this region as a type II region, then the y coordinate is bounded by $-2 \le y \le 2$, and the x coordinate is given by $0 \le x \le \sqrt{4-y^2}$. Therefore, interchanging the order of integration yields

$$\int_{-2}^2\int_0^{\sqrt{4-y^2}} f(x,y)\,dx\,dy.$$

In some problems, you will need to interchange the order of integration to solve the problem:

Example. (Example 5, Chapter 16.3) Evaluate the iterated integral

$$\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx.$$

If we try to integrate the inner integral as is, we are immediately stuck since we do not know of any formula for the indefinite integral, with respect to the variable y, of $\sin(y^2)$. Let us try interchanging the order of integration and see if that helps.

We begin (as always) by sketching the region of integration of this iterated integral. The outer bounds give the inequalities $0 \leq x \leq 1$, while the inner bound gives the inequalities $x \leq y \leq 1$. This is a triangle with vertices $(0,0),(0,1),(1,1)$, and we can re-express this triangle as a type II region with the inequalities $0 \leq y \leq 1$, $0 \leq x \leq y$. Therefore, the original iterated integral is equal to

$$\int_0^1 \int_0^y \sin(y^2)\,dx\,dy.$$

Notice that we can make progress on evaluating the inner integral, since it is now with respect to x, not y. This iterated integral equals

$$\int_0^1 \left(x \sin(y^2) \Big|_{x=0}^{x=y} \right) dy = \int_0^1 y \sin(y^2)\,dy.$$

Something wonderful happens – we can now evaluate the remaining integral using a simple u-substitution!

$$\int_0^1 y \sin(y^2)\,dy = -\frac{\cos(y^2)}{2}\Big|_0^1 = \frac{1}{2}(1 - \cos 1).$$