MATH 13 FINAL STUDY GUIDE, WINTER 2012

This is meant to be a quick reference guide for the topics you might want to know for the final. It probably isn't comprehensive, but should cover most of what we studied in this class. There are no examples, so be sure to consult your old homeworks, notes, or textbook for those.

1. Content from earlier classes (vectors, differentiation, etc.)

- Know about basic operations on vectors, such as addition, scalar multiplication, dot product, cross product, and their applications to geometric problems
- Know about properties of lines and planes in $\mathbb{R}^3$
- Know how to parameterize curves, compute arc length, and find tangent lines to curves
- Know about partial derivatives and their applications to computing tangent planes to surfaces
- Know about the gradient and directional derivatives

2. Double integration

- Integration over rectangles
- Integration over more general domains in the plane
- How to interchange order of integration (Fubini's Theorem)
- Polar coordinates
- Applications of double integration to finding area, mass of lamina, center of mass, etc.

3. Triple integration

- Integration over rectangular prisms
- Integration over more general three-dimensional solids
- How to interchange order of integration (Fubini's Theorem)
- Cylindrical coordinates
- Spherical coordinates
- Jacobian, change of variables formula (this covers both double and triple integrals)
- Applications of triple integration to finding volume, mass of solid, center of mass, etc.

4. Line integration, vector fields

- Parameterizing common curves (line segments, graphs of functions $y = f(x)$, circles, ellipses, etc)
- Calculating line integrals of scalar functions over curves
- Calculating line integrals of vector fields over curves
- The Fundamental Theorem of Calculus for line integrals
- The various properties of conservative vector fields

- How to apply the differential criterion $P_y = Q_x$ for being a conservative vector field, simply connected domains
- Applications of line integrals to calculating arc length, center of mass of a wire, work, etc.
- Green's Theorem

5. Surface integrals

- Parameterizing common surfaces (planes, cylinders, spheres, graphs of functions $z = f(x, y)$, etc)
- Calculating tangent plane and normal vectors from a parameterization for a surface (in particular, using the fundamental vector product)
- Calculating surface integrals of scalar functions over a surface
- Understanding what orientation on surfaces means
- Calculating surface integrals of vector fields over a surface
- Calculating divergence and curl of vector fields in $\mathbb{R}^3$
- How to apply the differential criterion $\nabla \times \mathbf{F} = \mathbf{0}$ for being a conservative vector field, simply connected domains
- The Divergence Theorem
- Stokes' Theorem
- Applications of surface integrals to finding surface area, etc

6. Formulas

A non-comprehensive collection of formulas from this class:

- Polar coordinates: $x = r\cos\theta, y = r\sin\theta$, if D is described by $\alpha \leq \theta \leq \beta, g_1(\theta) \leq r \leq g_2(\theta)$, then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta.$$

- Cylindrical coordinates: $x = r\cos\theta, y = r\sin\theta, z = z$, if E is described by $\alpha \leq \theta \leq \beta, r_1 \leq r \leq r_2, z_1 \leq z \leq z_2$ (z_i may be a function of r, θ, while r_i might be a function of θ), then

$$\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{r_1}^{r_2} \int_{z_1}^{z_2} f(r\cos\theta, r\sin\theta, z) r\,dz\,dr\,d\theta.$$

- Spherical coordinates: $x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$, if E is described by $\phi_1 \leq \phi \leq \phi_2, \ldots$, then

$$\iiint_E f(x,y,z)\,dV = \int_{\phi_1}^{\phi_2} \int_{\theta_1}^{\theta_2} \int_{\rho_1}^{\rho_2} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi\,d\rho\,d\theta\,d\phi.$$

- Jacobian: If T is a function from the uv plane to the xy plane, with $x = x(u,v), y = y(u,v)$, then the Jacobian of T is defined to be

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u.$$

The absolute value of the Jacobian represents a local area magnification factor. There is a corresponding formula for a transformation from uvw space to xyz space, involving a 3×3 determinant.

- Change of variables formula: if S is some region in the uv plane, and $R = T(S)$ (for example, if T is the polar change of coordinates, and S is the rectangle $0 \le r \le 1, 0 \le \theta \le 2\pi$, then $T(S)$ is the unit disc centered at the origin), then

$$\iint_{T(S)} f(x,y)\,dA = \iint_S f(x(u,v), y(u,v))|J(u,v)|\,du\,dv.$$

There is a corresponding formula for change of variables from uvw space to xyz space.

- Line integrals of scalar functions: If C is a curve parameterized by $\mathbf{r}(t) = \langle x(t), y(t)\rangle, a \le t \le b$, and $f(x,y)$ some function on C, then

$$\int_C f(x,y)\,ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\,dt = \int_a^b f(x(t),y(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt.$$

When $f(x,y) = 1$, this specializes to the formula for arc length of C. There is a corresponding formula for curves which lie in $\mathbb{R}^n$.

- Line integrals of vector fields: If C is a curve parameterized as above, and $\mathbf{F}(x,y)$ some vector field on C, then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_a^b \mathbf{F}(x(t),y(t))\cdot\langle x'(t), y'(t)\rangle\,dt.$$

If $\mathbf{F}$ represents a force, this line integral equals the work this force does on a particle which moves along the path C.

- The Fundamental Theorem of Calculus for Line Integrals: if $\mathbf{F} = \nabla f$ on a curve C parameterized as above, then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

- Conservative vector fields: their properties. A vector field $\mathbf{F}$ is defined to be conservative on a domain D if there exists a function f on D such that $\mathbf{F} = \nabla f$. This is equivalent to the following properties:
  - For any closed curve C in D, $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = 0$.
  - For any two curves C_1, C_2 in D which start and end at the same point, $\displaystyle\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$.

  If a vector field $\mathbf{F}\langle P, Q\rangle$ is conservative, then $P_y = Q_x$ in all of D. However, in general the converse is true only if D is simply connected. For $\mathbf{F}$ in $\mathbb{R}^3$, the condition $P_y = Q_x$ is replaced by $\nabla \times \mathbf{F} = \mathbf{0}$.

- Green's Theorem: If C is a simple closed curve in the plane, D its interior, $\mathbf{F} = \langle P, Q \rangle$ some C^1 vector field on D, and C is given the positive (counterclockwise) orientation, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D Q_x - P_y \, dA.$$

- Fundamental vector product: if a surface S is parameterized by $\mathbf{r}(u, v)$, the fundamental vector product is the cross product $\mathbf{r}_u \times \mathbf{r}_v$, and is in general a function of u, v. Its value at (u, v) is a normal vector to the surface S at the point $\mathbf{r}(u, v)$, and its length represents a local area magnification factor.
- Surface integral of a scalar function: if a surface S is parameterized by $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle, (u, v) \in D$, where D is some domain in the uv plane, then

$$\iint_S f(x, y, z) \, dS = \iint_D f(\mathbf{r}(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| \, dA = \iint_D f(x(u, v), y(u, v), z(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| \, dA.$$

When $f(x, y, z) = 1$, this specializes to the formula for the surface area of S.
- Surface integral of a vector field: if S is parameterized as above, given the orientation which points in the same direction as $\mathbf{r}_u \times \mathbf{r}_v$, and $\mathbf{F}(x, y, z)$ is a vector field on S, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA.$$

- The Divergence Theorem: if E is some solid, S the boundary of E with outward orientation, and $\mathbf{F}$ a C^1 vector field over all of E, then

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E \nabla \cdot \mathbf{F} \, dV.$$

- Stokes' Theorem: If S is some oriented surface, C its boundary with induced orientation from S, and $\mathbf{F}$ some C^1 vector field on S, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}.$$

7. Strategies for solving problems

A list of strategies we have learned for solving various types of problems.

- If you run into an iterated integral which you can't seem to evaluate, try switching the order of integration or changing coordinate systems.
- Polar coordinates are probably useful when dealing with circles, sectors of circles, annuli, or other geometric figures related to circles.
- Cylindrical coordinates are good when dealing with cylinders, paraboloids, or cones.
- Spherical coordinates are good when dealing with spheres or pieces of spheres.

- When calculating the line integral of a vector field, you can sometimes apply the Fundamental Theorem of Calculus for line integrals (if you can find a potential function for $\mathbf{F}$, which is not always possible or feasible) to skip parameterization of the curve C.
- You can show that $\mathbf{F}$ is conservative on D in a variety of ways. You can either explicitly construct a potential function via 'partial integration', or check that $P_y = Q_x$ if D is simply connected.
- You can show that $\mathbf{F}$ is not conservative on D in a variety of ways. You can either check that $P_y \neq Q_x$, or find a closed curve in C for which the integral of $\mathbf{F}$ along C is not equal to 0, or show that no potential function exists by trying to do partial integration and arriving at a contradiction.
- When calculating the line integral of a vector field over a simple closed curve C, sometimes Green's Theorem can simplify the calculation. This is especially true if C is polygonal, like a rectangle, or some other curve which has somewhat complicated parameterization.
- When calculating the surface integral of a vector field, you can sometimes skip a lot of steps if you find out that $\mathbf{F} \cdot \mathbf{n}$ is a constant on S and S is some geometric shape which you can easily calculate the area of.
- The Divergence Theorem is useful when you find a surface integral of a vector field which is complicated but whose divergence is simple, and/or if the surface S encloses a polyhedral solid.
- Stokes' Theorem is sometimes useful for calculating line integrals if $\nabla \times \mathbf{F}$ has a simple form (like a constant vector field), and you can sometimes use Stokes' Theorem to interchange the calculation of a surface integral $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S}$ of a vector field $\mathbf{F}$ which equals $\nabla \times \mathbf{G}$ for some $\mathbf{G}$ to the calculation of $\displaystyle\iint_{S'} \mathbf{F} \cdot d\mathbf{S}$, if S, S' have the same boundary curve.