Math 13 Winter 20, Practice Exam I
Elements of solution

(1) **Let f be a function of three variables x, y and z. For each of the following assertions, select the correct answer.**

(a) The integral
$$\int_0^{2t}\int_0^{x}\int_0^{xz} f(x,y,z)\,dy\,dz\,dx$$

☐ is a constant.

☒ is a function of t.

☐ is a function of y.

☐ is not well-defined.

☐ none of the above.

(b) The integral
$$\int_0^{1}\int_x^{0}\int_0^{xz} f(x,y,z)\,dy\,dx\,dz.$$

☐ is a constant.

☐ is a function of x and z.

☐ is a function of y.

☒ is not well-defined.

☐ none of the above.

(c) The integral
$$\int_0^{1}\int_0^{2} f(x,y,z)\,dx\,dz.$$

☐ is a constant.

☐ is a function of x and z.

☒ is a function of y.

☐ is not well-defined.

☐ none of the above.

(2) **Calculate the double integral**

$$\iint_{\mathcal{R}} \left(\frac{x}{y} + \frac{y}{x}\right)\, dA$$

where $\mathcal{R} = [1,4] \times [1,2]$**.**

$$\begin{aligned}
\iint_{\mathcal{R}} \left(\frac{x}{y} + \frac{y}{x}\right)\, dA &= \int_{y=1}^{2}\int_{x=1}^{4} \left(\frac{x}{y} + \frac{y}{x}\right) dx\, dy \\
&= \int_{y=1}^{2} \left[\frac{1}{y}\frac{x^2}{2} + y\ln x\right]_{x=1}^{4} dy \\
&= \int_{y=1}^{2} \left(\frac{15}{2y} + 2y\ln 2\right) dy \\
&= \left[\frac{15}{2}\ln y + y^2 \ln 2\right]_{y=1}^{2} = \boxed{\frac{21\ln 2}{2}}.
\end{aligned}$$

(3) **Evaluate the integral** $\int_0^1 \int_{3y}^{3} e^{x^2}\, dx\, dy$**.**

No antiderivative of e^{x^2} can be expressed in terms of ordinary functions so we reverse the order of integration:

$$\begin{aligned}
\int_0^1 \int_{3y}^{3} e^{x^2}\, dx\, dy &= \int_0^3 \int_0^{\frac{x}{3}} e^{x^2}\, dy\, dx \\
&= \int_0^3 \left[y e^{x^2}\right]_0^{\frac{x}{3}} dx \\
&= \int_0^3 \frac{x}{3} e^{x^2}\, dx \\
&= \frac{1}{3}\int_{u=0}^{9} e^u \frac{du}{2} = \boxed{\frac{e^9 - 1}{6}}.
\end{aligned}$$

(4) **Calculate the triple integral** $\int_0^{\sqrt{\pi}} \int_0^x \int_0^{xz} x^2 \cos(y)\, dy\, dz\, dx$**.**

$$\begin{aligned}
\int_0^{\sqrt{\pi}} \int_0^x \int_0^{xz} x^2 \cos(y)\, dy\, dz\, dx &= \int_0^{\sqrt{\pi}} \int_0^x x^2 \sin(xz)\, dz\, dx \\
&= \int_0^{\sqrt{\pi}} \left[-x\cos(xz)\right]_{z=0}^{x} dx \\
&= \int_0^{\sqrt{\pi}} \left(-x\cos(x^2) + x\right) dx \\
&= -\int_0^{\pi} \cos(u)\frac{du}{2} + \frac{\pi}{2} = \boxed{\frac{\pi}{2}}.
\end{aligned}$$

(5) **Find the volume of the solid that lies below the surface $z^2 = 9x^2 + 9y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2y$.**

The base of the cylinder is the circle in the xy-plane with center $(0, 1)$ and radius 1. In polar coordinates, it is described by $r = 2\sin\theta$ for $0 \leq \theta \leq \pi$. Since z is assumed non-negative, the upper boundary of the solid has equation $z = 3r$ in cylindrical coordinates.

It follows that

$$\begin{aligned} \text{Volume} &= \int_0^\pi \int_0^{2\sin\theta} \int_0^{3r} r\,dz\,dr\,d\theta \\ &= \int_0^\pi \int_0^{2\sin\theta} 3r^2\,dr\,d\theta \\ &= 8\int_0^\pi \sin^3\theta\,d\theta = \boxed{\frac{32}{3}}. \end{aligned}$$

(6) **The volume of the solid bounded by the surfaces with equations**

$$y = x^2 + \frac{z^2}{4} \qquad \textbf{and} \qquad y = 5 - 4x^2 - z^2$$

is given by an integral of the form

$$\int_a^b \int_{u_1(z)}^{u_2(z)} \int_{x^2+\frac{z^2}{4}}^{5-4x^2-z^2} dy\,dx\,dz.$$

Determine $u_1(z)$, $u_2(z)$, a and b.

To identify the projection onto the xz-plane of the solid, we determine the intersection curve of the paraboloids:

$$y = x^2 + \frac{z^2}{4} = 5 - 4x^2 - z^2$$

implies $x^2 + \frac{z^2}{4} = 1$, which is the equation of an ellipse in the $y = 0$ plane. It follows that

$$a = -2 \quad , \quad b = 2 \quad , \quad u_1(z) = -\sqrt{1 - \frac{z^2}{4}} \quad , \quad u_2(z) = \sqrt{1 - \frac{z^2}{4}}.$$

(7) **Let $\mathcal{W}$ be the solid region lying:**

- **inside the cone $z = \sqrt{x^2 + y^2}$**
- **inside the sphere $x^2 + y^2 + z^2 = 5$**
- **outside the sphere $x^2 + y^2 + z^2 = 3$**
- **in the second octant: $\boldsymbol{x \leq 0}$, $y \geq 0$, $z \geq 0$.**

Convert the integral $\iiint_{\mathcal{W}} x\, dV$ into a triple integral in spherical coordinates.

In spherical coordinates, $\mathcal{W}$ consists of the points (ρ, θ, φ) such that

$$\sqrt{3} \leq \rho \leq \sqrt{5} \quad , \quad \frac{\pi}{2} \leq \theta \leq \pi \quad , \quad 0 \leq \varphi \leq \frac{\pi}{4}$$

Therefore,

$$\iiint_{\mathcal{W}} x\, dV = \int_{\frac{\pi}{2}}^{\pi} \int_{0}^{\frac{\pi}{4}} \int_{\sqrt{3}}^{\sqrt{5}} \rho^3 \sin^2 \varphi \cos \theta \, d\rho \, d\varphi \, d\theta.$$

(8) **Express the volume of the region bounded by the surfaces with equations**

- $z = x^2 - 1$
- $x = y^2$
- $z = 0$

by a triple integral in the prescribed order:

(a) $dz\, dy\, dx$
(b) $dy\, dx\, dz$

$$\text{Volume} = \int_{x=0}^{1} \int_{y=-\sqrt{x}}^{\sqrt{x}} \int_{z=x^2-1}^{0} dz\, dy\, dx = \int_{z=-1}^{0} \int_{x=0}^{\sqrt{z+1}} \int_{y=-\sqrt{x}}^{\sqrt{x}} dy\, dx\, dz.$$

(9) **A lamina has the shape of a half-disk of radius 2, centered at the origin and lying in the upper-half plane. Its density function is $\delta(x, y) = k\sqrt{x^2 + y^2}$, where k is a constant. The center of mass of the lamina has coordinates $(0, y_0)$. Determine y_0.**

Observe that the density only depends on the distance to the origin: $\delta(r, \theta) = kr$. The total mass of the lamina is

$$M = \int_0^{\pi} \int_0^2 kr^2 \, dr \, d\theta = \frac{8k\pi}{3}.$$

The vertical moment is

$$M_x = \int_0^{\pi} \int_0^2 kr^3 \sin \theta \, dr \, d\theta = 8k.$$

It follows that

$$y_0 = \frac{M_x}{M} = \frac{3}{\pi}.$$