# Math 13 Winter 20, Practice Exam II
# Elements of solution

(1) **For each of the following assertions, select the correct ending.**

(a) Let $\mathbb{D}$ be the disk with radius 1 and center at the origin in the $uv$-plane and $G$ the mapping defined by

$$G(u, v) = (u + 2v, -u + 4v).$$

The area of $G(\mathbb{D})$ is...

☒ $6\pi$.

☐ $\frac{\pi}{6}$.

☐ $\pi$.

☐ $2\pi$.

☐ $\frac{\pi}{2}$.

☐ none of the above.

(b) Let $f(x, y, z) = y^3$. Then $\text{grad}\,(\text{div}\,(\text{grad}\, f)) = \ldots$

☒ $6\mathbf{j}$.

☐ 6.

☐ $6y$.

☐ $\langle 0, 3y^2, 0 \rangle$.

☐ none of the above.

(c) Let $\mathbf{F}$ be a vector field. Then $\text{grad}\,(\text{curl}\,\mathbf{F})$ is...

☐ a potential for $\mathbf{F}$.

☐ the divergence of $\mathbf{F}$.

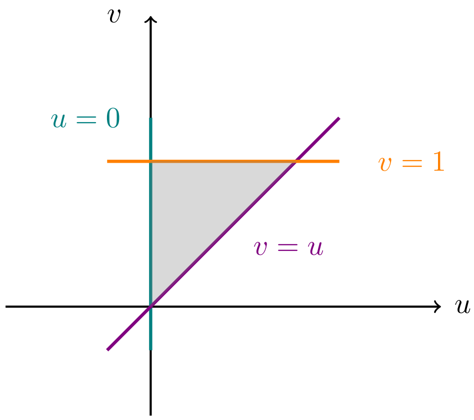
☐ a vector field.

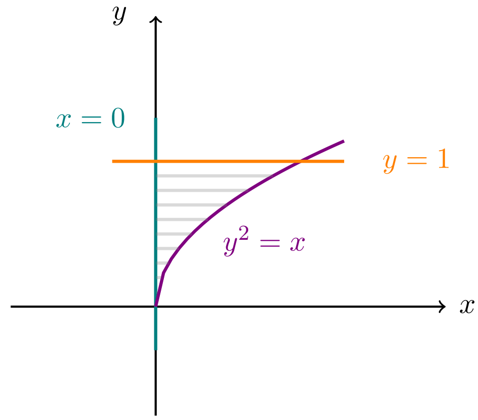
☒ not well-defined.

☐ none of the above.

(2) **Determine the image of the triangular region with vertices $(0,0)$, $(1,1)$ and $(0,1)$ under the transformation $G(u,v) = (u^2, v)$.**

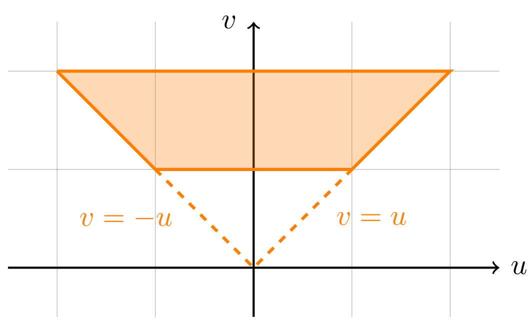
The transformation is given by $x = u^2$ and $y = v$ and the original region is bounded by the lines with equations $u = 0$, $v = 1$ and $u = v$ which transform under $G$ to $x = 0$, $y = 1$ and $y^2 = x$.




(3) **The mapping $G(u,v) = \left(\frac{1}{2}(u+v), \frac{1}{2}(u-v)\right)$ transforms the trapezoidal region $\mathcal{S}$ with vertices $(-2,2)$, $(2,2)$, $(1,1)$ and $(-1,1)$ in the $uv$-plane into the trapezoidal region $\mathcal{R}$ with vertices $(1,0)$, $(2,0)$, $(0,-2)$ and $(0,-1)$ in the $xy$-plane.**

**Use this change of variables to evaluate the integral $\iint_{\mathcal{R}} e^{\frac{x+y}{x-y}}\, dA$.**




Observing that $x + y = u$ and $x - y = v$ and that $\text{Jac}(G) = -\frac{1}{2}$, we obtain

$$\begin{aligned}
\iint_{\mathcal{R}} e^{\frac{x+y}{x-y}}\, dA &= \frac{1}{2} \iint_{\mathcal{S}} e^{\frac{u}{v}}\, du\, dv \\
&= \frac{1}{2} \int_1^2 \int_{-v}^{v} e^{\frac{u}{v}}\, du\, dv \\
&= \frac{1}{2} \int_1^2 v \left(e - \frac{1}{e}\right) dv \\
&= \boxed{\frac{3}{4}\left(e - \frac{1}{e}\right)}.
\end{aligned}$$

(4) **The volume of the solid ellipsoid $\mathcal{E}$ : $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \leq 1$ is given by:**

$$\mathrm{Vol}(\mathcal{E}) = \iiint_{\mathcal{E}} 1 \, dV.$$

**Use one of the following changes of variables and the fact that the ball with center at the origin and radius 1 has volume $\frac{4}{3}\pi$ to determine $\mathrm{Vol}(\mathcal{E})$.**

$$G_1(u,v,w) = \left(\frac{u}{a}, \frac{v}{b}, \frac{w}{c}\right)$$
$$G_2(u,v,w) = (au, bv, cw)$$
$$G_3(u,v,w) = (\sqrt{u}, \sqrt{v}, \sqrt{w})$$

Let us use $G_2$ and set $x = au$, $y = bv$ and $z = cw$. Then,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \quad \Leftrightarrow \quad u^2 + v^2 + w^2 \leq 1$$

and the Change of Variables Formula gives

$$\mathrm{Vol}(\mathcal{E}) = \iiint_{\mathcal{B}} 1 \cdot |\mathrm{Jac}(G_2)| \, du \, dv \, dw$$

where $\mathcal{B}$ denotes the unit ball in the $uvw$-space and, assuming $a, b, c > 0$,

$$\mathrm{Jac}(G_2) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc.$$

It follows that

$$\mathrm{Vol}(\mathcal{E}) = abc \iiint_{\mathcal{B}} 1 \, du \, dv \, dw = abc \cdot \mathrm{Vol}(\mathcal{B}) = \boxed{\frac{4abc}{3}\pi}.$$

(5) **Evaluate the integral**

$$\int_{\mathcal{C}} 18y^3 \, ds$$

**where $\mathcal{C}$ is the curve in the plane parameterized by $x(t) = t^3$, $y(t) = t$ with $0 \leq t \leq 1$.**

By definition,

$$\begin{aligned} \int_{\mathcal{C}} 18y^3 \, ds &= 18 \int_0^1 y(t)^3 \sqrt{x'(t)^2 + y'(t)^2} \, dt \\ &= 18 \int_0^1 t^3 \sqrt{9t^4 + 1} \, dt. \end{aligned}$$

The change of variables $u = 9t^4 + 1$ with $du = 36t^3 \, dt$ further leads to

$$\int_{\mathcal{C}} 18y^3 \, ds = \frac{1}{2} \int_1^{10} \sqrt{u} \, du = \boxed{\frac{10\sqrt{10} - 1}{3}}.$$

(6) **Evaluate the integral** $\int_{\mathcal{L}} xe^{yz}\,ds$ **where** $\mathcal{L}$ **is the segment from** $(0,0,0)$ **to** $(1,2,3)$**.**

We can parametrize $\mathcal{L}$ by $\mathbf{r}(t) = \langle t, 2t, 3t\rangle$ for $0 \le t \le 1$ so that $ds = \sqrt{14}\,dt$ and

$$\int_{\mathcal{L}} xe^{yz}\,ds = \sqrt{14}\int_0^1 te^{6t^2}\,dt = \frac{\sqrt{14}}{12}\int_0^6 e^u\,du = \boxed{\frac{\sqrt{14}(e^6-1)}{12}}.$$

(7) **Evaluate the integral of the vector field**

$$\mathbf{F}(x,y,z) = \left(x+y+\frac{z}{4}\right)\mathbf{i} + (y-x^3)\mathbf{j} + \ln\left(\frac{x+z}{y+1}\right)\mathbf{k}$$

**along the curve** $\Gamma$ **given by** $\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j} + 4\mathbf{k}$ **for** $0 \le t \le 1$**.**

$$\begin{aligned}\int_\Gamma \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(x(t),y(t),z(t))\cdot \mathbf{r}'(t)\,dt \\ &= \int_0^1 \left\langle t+t^3+1, 0, \ln\left(\frac{t+4}{t^3+1}\right)\right\rangle \cdot \langle 1, 3t^2, 0\rangle\,dt \\ &= \int_0^1 (t+t^3+1)\,dt = \boxed{\frac{7}{4}}.\end{aligned}$$

(8) **The vector field** $\mathbf{F}(x,y) = (3+2xy)\mathbf{i} + (x^2-3y^2)\mathbf{j}$ **is conservative. Find a potential.**

A potential for $\mathbf{F}$ is a function $f(x,y)$ satisfying

$$\frac{\partial f}{\partial x}(x,y) = 3+2xy \qquad \text{and} \qquad \frac{\partial f}{\partial y}(x,y) = x^2-3y^2.$$

The first identity implies that

$$f(x,y) = 3x + x^2y + K(y)$$

where $K(y)$ is a function of $y$ only. Differentiating this identity with respect to $y$ gives

$$\frac{\partial f}{\partial y}(x,y) = x^2 + K'(y)$$

and the second potential relation implies that $K'(y) = -3y^2$, so that $K(y) = -y^3 + C$ where $C$ is any real constant. As a conclusion, any potential for $\mathbf{F}$ is of the form

$$\boxed{3x + x^2y - y^3 + C}.$$

(9) **Let** $f(x,y) = xe^y$ **and** $\mathbf{F} = \nabla f$**. Evaluate** $\int_\Gamma \mathbf{F}\cdot d\mathbf{r}$**, where** $\Gamma$ **is given by**

$$\mathbf{r}(t) = te^t\mathbf{i} + \sqrt{1+3t}\,\mathbf{j}$$

**for** $0 \le t \le 1$**.**

The Fundamental Theorem of Calculus for line integrals gives

$$\int_\Gamma \mathbf{F}\cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(e,2) - f(0,1) = \boxed{e^3}.$$