

Math 13 Winter 20, Practice Exam III

Elements of solution

(1) For each of the following assertions, select the correct ending.

(a) If $f(x, y)$ is a function, then $\int_0^2 \int_0^x f(x, y) dy dx = \dots$

$\int_0^2 \int_0^2 f(x, y) dx dy.$

$\int_0^2 \int_0^x f(x, y) dx dy.$

$\int_0^2 \int_0^y f(x, y) dx dy.$

$\int_0^2 \int_y^2 f(x, y) dx dy.$

$\int_0^x \int_0^2 f(x, y) dx dy.$

none of the above.

(b) A potential for the vector field $\mathbf{F}(x, y, z) = \sin(yz^2)\mathbf{i} + xz^2 \cos(yz^2)\mathbf{j} + 2xyz \cos(yz^2)\mathbf{k}$ is...

$\langle 0, -xz^4 \sin(yz^2), -4xy^2z^2 \sin(yz^2) \rangle.$

$\langle \sin(yz^2), xz^2 \cos(yz^2), 2xyz \cos(yz^2) \rangle.$

$x \sin(yz^2) + 2.$

$-xz^2(z^2 + 4y^2) \sin(yz^2).$

none of the above.

(c) Let $\mathbf{F}(x, y, z)$ be a vector field defined on an open, connected and simply connected domain. Then \mathbf{F} is conservative if and only if...

$\mathbf{F} = \nabla f$ for some scalar function $f(x, y, z).$

$\nabla \times \mathbf{F} = \mathbf{0}.$

$\oint_{\Gamma} \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve Γ in the domain of $\mathbf{F}.$

all of the above.

none of the above.

(d) **The Fundamental Theorem of line integrals...**

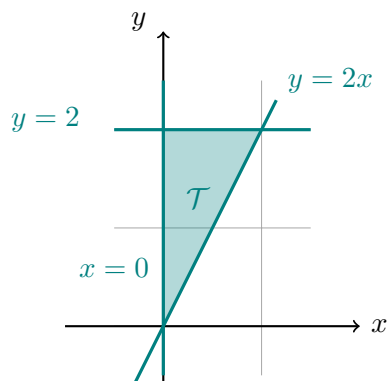
- applies to scalar functions.
- only applies to conservative vector fields.
- applies to all vector fields.
- only applies to plane curves, not to curves in 3-dimensional space.
- none of the above.

(2) Calculate $\iint_{\mathcal{R}} \frac{x}{1+xy} dA$ where $\mathcal{R} = [0, 1] \times [0, 1]$.

$$\begin{aligned} \iint_{\mathcal{R}} \frac{x}{1+xy} dA &= \int_{x=0}^1 \int_{y=0}^1 \frac{x}{1+xy} dy dx \\ &= \int_{x=0}^1 [\ln(1+xy)]_{y=0}^1 dx \\ &= \int_{x=0}^1 \ln(1+x) dx \\ &= \int_1^2 \ln(u) du \\ &= [u \ln(u) - u]_1^2 = \boxed{2 \ln(2) - 1}. \end{aligned}$$

(3) Find the volume of the solid bounded by the planes $x = 0$, $y = 2$, $z = 0$, $y = 2x$ and the surface $z = y^2$.

The solid consists of the part of the solid cylinder with base \mathcal{T} pictured below and comprised between the xy -plane and the surface $z = y^2$.

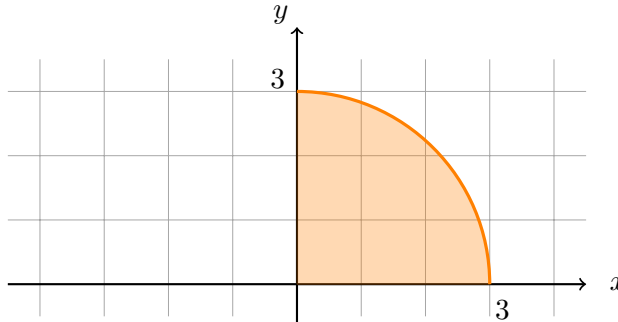


so the volume is equal to

$$\iint_{\mathcal{T}} y^2 dA = \int_0^1 \int_{2x}^2 y^2 dy dx = \frac{8}{3} \int_0^1 (1 - x^3) dx = \boxed{2}.$$

(4) Consider the iterated integral $\mathcal{I} = \int_0^3 \int_0^{\sqrt{9-x^2}} (x^2 + y^2) dy dx$.

(a) Sketch the domain of integration in the xy -plane.



(b) Evaluate \mathcal{I} .

Using polar coordinates, $\mathcal{I} = \int_0^{\frac{\pi}{2}} \int_0^3 r^2 \cdot r dr d\theta = \boxed{\frac{81\pi}{8}}$.

(5) Let \mathcal{W} be a solid with density $\delta(x, y, z)$. Match the following quantities with their expression:

A. Total mass of \mathcal{W}

B. y -coordinate of the center of mass

C. Mean value of the density

$\frac{\iiint_{\mathcal{W}} x dV}{\iiint_{\mathcal{W}} dV}$

C $\frac{\iiint_{\mathcal{W}} \delta(x, y, z) dV}{\iiint_{\mathcal{W}} dV}$

$\frac{\iiint_{\mathcal{W}} y dV}{\iiint_{\mathcal{W}} dV}$

$\frac{\iiint_{\mathcal{W}} x \delta(x, y, z) dV}{\iiint_{\mathcal{W}} \delta(x, y, z) dV}$

$\frac{\iiint_{\mathcal{W}} z dV}{\iiint_{\mathcal{W}} dV}$

B $\frac{\iiint_{\mathcal{W}} y \delta(x, y, z) dV}{\iiint_{\mathcal{W}} \delta(x, y, z) dV}$

$\iiint_{\mathcal{W}} 1 dV$

$\frac{\iiint_{\mathcal{W}} z \delta(x, y, z) dV}{\iiint_{\mathcal{W}} \delta(x, y, z) dV}$

A $\iiint_{\mathcal{W}} \delta(x, y, z) dV$

- (6) Let \mathcal{D} be the domain bounded by the curves $y = x$, $y = 3x$, $xy = 1$ and $xy = 3$ in the first quadrant. What does the integral $\iint_{\mathcal{D}} xy \, dA$ become under the change of variables $x = \frac{u}{v}$, $y = v$?

$\int_1^3 \int_1^3 u \, dv \, du.$

$\int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} u \, dv \, du.$

$\int_1^3 \int_{u^2}^{3u^2} u \, dv \, du.$

$\int_1^3 \int_1^3 \frac{u}{v} \, dv \, du.$

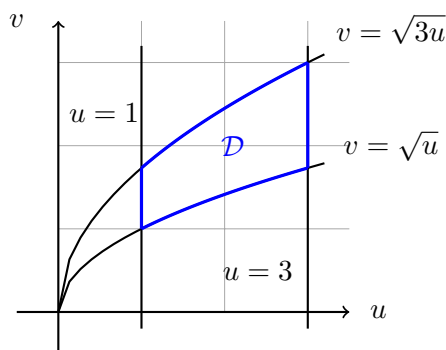
$\int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} \frac{u}{v} \, dv \, du.$

$\int_1^3 \int_{u^2}^{3u^2} \frac{u}{v} \, dv \, du.$

Under the change of variables, the boundary curves
 $y = x$, $y = 3x$, $xy = 1$ and $xy = 3$
 respectively become

$$v^2 = u \quad , \quad v^2 = 3u \quad , \quad u = 1 \quad \text{and} \quad u = 3$$

giving the domain of integration \mathcal{D} below.



The Jacobian of the transformation is $\left| \begin{array}{cc} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{array} \right| = \frac{1}{v}.$

- (7) Let Γ be the curve parametrized by $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + e^t\mathbf{k}$ for $0 \leq t \leq 1$.

- (a) Determine the starting point and the end point of Γ .

Setting $t = 0$ gives $P = (0, 0, 1)$ and $t = 1$ leads to $Q = (1, 2, e)$.

- (b) Consider the function $f(x, y, z) = x\sqrt{y}\ln(1 + x - y + z)$ and let $\mathbf{F} = \nabla f$.

Evaluate $\int_{\Gamma} \mathbf{F} \cdot d\mathbf{r}$.

By the Fundamental Theorem for line integrals, $\int_{\Gamma} \nabla f \cdot d\mathbf{r} = f(Q) - f(P) = \boxed{\sqrt{2}}.$

- (8) Let Σ be the part of the sphere with equation $x^2 + y^2 + z^2 = 1$ in the $x < 0$ region. The integral $\iint_{\Sigma} x \, dS$ is equal to...

$\int_0^{\pi} \int_0^{\pi} \cos \theta \sin \varphi \, d\theta \, d\varphi.$

Using spherical coordinates, we get

$$x = \cos \theta \sin \varphi$$

$\int_0^{\pi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos \theta \sin \varphi \, d\theta \, d\varphi.$

and the magnitude of the normal vector is

$$\|N(\theta, \varphi)\| = \sin \varphi$$

$\int_0^{\pi} \int_0^{2\pi} \cos \theta \sin \varphi \, d\theta \, d\varphi.$

so the integrand must be

$$\cos \theta \sin^2 \varphi.$$

$\int_0^{\pi} \int_0^{2\pi} \cos \theta \sin^2 \varphi \, d\theta \, d\varphi.$

As for the range of the parameters, there is no restriction on z so

$$0 \leq \varphi \leq \pi$$

$\int_0^{\pi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos \theta \sin^2 \varphi \, d\theta \, d\varphi.$

and the condition $x < 0$ leads to

$$\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}.$$

$\int_0^{\pi} \int_0^{\pi} \cos \theta \sin^2 \varphi \, d\theta \, d\varphi.$

$\int_0^{\pi} \int_0^{2\pi} \int_0^1 \cos \theta \sin^2 \varphi \, \rho \, d\rho \, d\theta \, d\varphi.$

$\int_0^{\pi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^1 \cos \theta \sin^2 \varphi \, \rho \, d\rho \, d\theta \, d\varphi.$

$\int_0^{\pi} \int_0^{\pi} \int_0^1 \cos \theta \sin^2 \varphi \, \rho \, d\rho \, d\theta \, d\varphi.$

- (9) Let Σ be the surface parametrized by $S(u, v) = (u^2, v^2, u + 2v)$ for $0 \leq u \leq 2$ and $0 \leq v \leq 2$. Find an equation for the plane tangent to Σ at $(1, 1, 3)$.

Note that $(1, 1, 3) = S(1, 1)$ so a normal vector \mathbf{N} for the tangent plane can be obtained by taking the crossed product $\mathbf{T}_u(1, 1) \times \mathbf{T}_v(1, 1)$ where

$$\mathbf{T}_u(u, v) = \langle 2u, 0, 1 \rangle \quad \text{and} \quad \mathbf{T}_v(u, v) = \langle 0, 2v, 2 \rangle.$$

Therefore, we choose

$$\mathbf{N} = \langle 2, 0, 1 \rangle \times \langle 0, 2, 2 \rangle = \langle -2, -4, 4 \rangle = -2 \cdot \langle 1, 2, -2 \rangle$$

and the equation is of the form $x + 2y - 2z + d = 0$. Finally we determine d so that $(1, 1, 3)$ satisfies the equation:

$$\boxed{x + 2y - 2z + 3 = 0}.$$

(10) Evaluate $\int_{\Sigma} \mathbf{F} \cdot d\mathbf{S}$ where

$$\mathbf{F}(x, y, z) = \langle xy, 4x^2, yz \rangle$$

and Σ is the surface $z = xe^y$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$, oriented in the positive x -direction.

A parametrization for Σ is

$$S(u, v) = (u, v, ue^v) \quad , \quad 0 \leq u \leq 1 \quad , \quad 0 \leq v \leq 1$$

with tangent vectors $\mathbf{T}_u = \langle 1, 0, e^v \rangle$ and $\mathbf{T}_v = \langle 0, 1, ue^v \rangle$ so that

$$\mathbf{T}_u \times \mathbf{T}_v = \langle -e^v, -ue^v, 1 \rangle.$$

Given the orientation on Σ , we choose $\mathbf{N}(u, v) = -\mathbf{T}_u \times \mathbf{T}_v$ and since

$$\mathbf{F}(S(u, v)) = \langle uv, 4u^2, uve^v \rangle$$

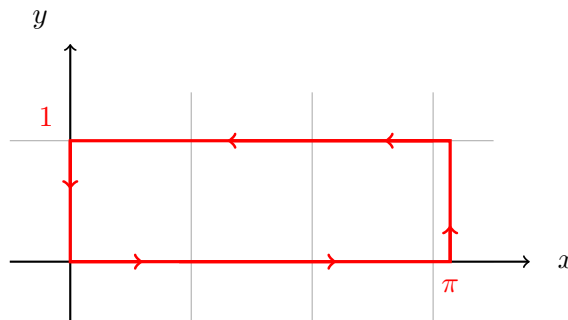
we get

$$\begin{aligned} \int_{\Sigma} \mathbf{F} \cdot d\mathbf{S} &= \int_0^1 \int_0^1 \langle uv, 4u^2, uve^v \rangle \cdot \langle e^v, ue^v, -1 \rangle \, du \, dv \\ &= \int_0^1 \int_0^1 4u^3 e^v \, du \, dv \\ &= 4 \int_0^1 u^3 \, du \int_0^1 e^v \, dv = \boxed{e-1}. \end{aligned}$$

(11) Calculate the circulation of the vector field

$$\mathbf{F}(x, y) = \langle y^3 + 2xy + 2y - e^{-x^2}, x^2 + 3xy^2 + 5x + \cos(\sqrt{y}) \rangle$$

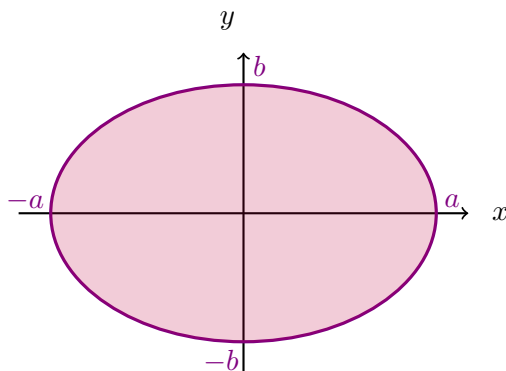
along the closed rectangular path Γ drawn below.



By Green's Theorem,

$$\begin{aligned} \oint_{\Gamma} \mathbf{F} \cdot d\mathbf{x} &= \iint_{[0,\pi] \times [0,1]} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA \\ &= \int_0^1 \int_0^{\pi} (2x + 3y^2 + 5 - (3y^2 + 2x + 2)) \, dx \, dy \\ &= 3 \int_0^1 \int_0^{\pi} dx \, dy = \boxed{3\pi}. \end{aligned}$$

- (12) Use Green's Theorem to calculate the surface of the elliptic domain \mathcal{E} with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$.



Parameterizing the ellipse by $\begin{cases} x(t) = a \cos t \\ y(t) = b \sin t \end{cases}$ for $0 \leq t \leq 2\pi$, it is a consequence of Green's Theorem that

$$\begin{aligned} \text{Area}(\mathcal{E}) &= \frac{1}{2} \oint_{\partial\mathcal{E}} x \, dy - y \, dx \\ &= \frac{1}{2} \int_0^{2\pi} (a \cos t \cdot b \cos t - b \sin t \cdot (-a \sin t)) \, dt \\ &= \frac{ab}{2} \int_0^{2\pi} dt = \boxed{\pi ab}. \end{aligned}$$

- (13) Let \mathbb{S} be the hemisphere $x^2 + y^2 + z^2 = 9$ in the $z \geq 0$ region, oriented upward and $\mathbf{F}(x, y, z) = 2x \cos z \mathbf{i} + e^x \sin z \mathbf{j} + xe^y \mathbf{k}$. Evaluate $\iint_{\mathbb{S}} \text{curl } \mathbf{F} \cdot d\mathbf{S}$.

By Stokes' Theorem, $\iint_{\mathbb{S}} \text{curl } \mathbf{F} \cdot d\mathbf{S} = \oint_{\Gamma} \mathbf{F} \cdot d\mathbf{r}$ where Γ is the circle parametrized by $\mathbf{r}(t) = \langle 3 \cos t, 3 \sin t, 0 \rangle$, $0 \leq t \leq 2\pi$.

Since

$$\mathbf{F}(\mathbf{r}(t)) = \langle 6 \cos t, 0, \cos t e^{\sin t} \rangle \quad \text{and} \quad \mathbf{r}'(t) = \langle -3 \sin t, 3 \cos t, 0 \rangle,$$

we get

$$\iint_{\mathbb{S}} \text{curl } \mathbf{F} \cdot d\mathbf{S} = \boxed{-18 \int_0^{2\pi} \cos t \sin t \, dt = 0}.$$

(14) Let Σ denote the part of the paraboloid $y = x^2 + z^2$ with $0 \leq y \leq 1$. Consider

$$\mathbf{F}(x, y, z) = \langle z\sqrt{7} + y + e^{x^2}, xy + \sin \sqrt{y}, 0 \rangle \quad \text{and} \quad \mathbf{G} = \text{curl } \mathbf{F}.$$

Evaluate $\iint_{\Sigma} \mathbf{G} \cdot d\mathbf{S}$, using normal vectors pointing in the positive y -direction.

By Stokes' Theorem, the flux of $\text{curl } \mathbf{F}$ is the same across any surface having the same compatibly oriented boundary as Σ . This boundary is the circle centered at $(0, 1, 0)$ with radius 1. It is also the boundary of the disk \mathcal{D} centered at $(0, 1, 0)$ with radius 1 in the $y = 1$ plane. A parametrization for \mathcal{D} is

$$S(r, \theta) = (r \cos \theta, 1, r \sin \theta) \quad , \quad 0 \leq r \leq 1 \quad , \quad 0 \leq \theta \leq 2\pi$$

with normal vector $\langle 0, r, 0 \rangle$ compatible with the orientation of the circle inherited from that of Σ . Since $\text{curl } \mathbf{F}(x, y, z) = \langle 0, \sqrt{7}, y - 1 \rangle$, we get

$$\mathbf{G}(S(r, \theta)) = \langle 0, \sqrt{7}, 0 \rangle$$

and

$$\begin{aligned} \iint_{\Sigma} \mathbf{G} \cdot d\mathbf{S} &= \iint_{\mathcal{D}} \mathbf{G} \cdot d\mathbf{S} \\ &= \int_0^{2\pi} \int_0^1 \langle 0, \sqrt{7}, 0 \rangle \cdot \langle 0, r, 0 \rangle \, dr \, d\theta \\ &= 2\pi\sqrt{7} \int_0^1 r \, dr \\ &= \boxed{\pi\sqrt{7}}. \end{aligned}$$

(15) Let \mathbf{F} be a vector field defined on a simple solid region \mathcal{W} with boundary the closed surface \mathcal{S} . Then, $\iint_{\mathcal{S}} \text{curl } \mathbf{F} \cdot d\mathbf{S}$ is...

a vector.

$\iiint_{\mathcal{W}} \text{div } \mathbf{F} \, dV$.

0 by the Divergence Theorem.

0 by Stokes Theorem.

not well-defined.

none of the above.

One can apply Stokes' Theorem and notice that $\partial\mathcal{S}$ is empty so

$$\iint_{\mathcal{S}} \text{curl } \mathbf{F} \cdot d\mathbf{S} = \oint_{\emptyset} \mathbf{F} \cdot d\mathbf{r} = 0$$

or the Divergence Theorem, leading to

$$\iint_{\mathcal{S}} \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \underbrace{\text{div}(\text{curl } \mathbf{F})}_{=0} = 0.$$