Math 13 Winter 20, Practice Exam III

Name: ______________________________

INSTRUCTIONS

This is a closed book, closed notes exam.

There are 15 problems. You have 3 hours.

Use of calculators is not permitted.

On each question you must show your work. No credit is given for solutions without supporting calculations.

GOOD LUCK!

(1) (*12 points*) For each of the following assertions, select the correct ending.

(a) If $f(x,y)$ is a function, then $\displaystyle\int_0^2 \int_0^x f(x,y)\,dy\,dx = \ldots$

☐ $\displaystyle\int_0^2 \int_0^2 f(x,y)\,dx\,dy.$

☐ $\displaystyle\int_0^2 \int_0^x f(x,y)\,dx\,dy.$

☐ $\displaystyle\int_0^2 \int_0^y f(x,y)\,dx\,dy.$

☐ $\displaystyle\int_0^2 \int_y^2 f(x,y)\,dx\,dy.$

☐ $\displaystyle\int_0^x \int_0^2 f(x,y)\,dx\,dy.$

☐ none of the above.

(b) A potential for the vector field $\mathbf{F}(x,y,z) = \sin(yz^2)\mathbf{i} + xz^2\cos(yz^2)\mathbf{j} + 2xyz\cos(yz^2)\mathbf{k}$ is...

☐ $\langle 0\,,\, -xz^4\sin(yz^2)\,,\, -4xy^2z^2\sin(yz^2)\rangle.$

☐ $\langle \sin(yz^2)\,,\, xz^2\cos(yz^2)\,,\, 2xyz\cos(yz^2)\rangle.$

☐ $x\sin(yz^2) + 2.$

☐ $-xz^2(z^2+4y^2)\sin(yz^2).$

☐ none of the above.

(c) Let $\mathbf{F}(x, y, z)$ be a vector field defined on an open, connected and simply connected domain. Then $\mathbf{F}$ is conservative if and only if...

☐ $\mathbf{F} = \nabla f$ for some scalar function $f(x, y, z)$.

☐ $\nabla \times \mathbf{F} = \mathbf{0}$.

☐ $\displaystyle\oint_{\Gamma} \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve Γ in the domain of $\mathbf{F}$.

☐ all of the above.

☐ none of the above.

(d) The Fundamental Theorem of line integrals...

☐ applies to scalar functions.

☐ only applies to conservative vector fields.

☐ applies to all vector fields.

☐ only applies to plane curves, not to curves in 3-dimensional space.

☐ none of the above.

(2) (*7 points*) Calculate

$$\iint_{\mathcal{R}} \frac{x}{1+xy}\, dA$$

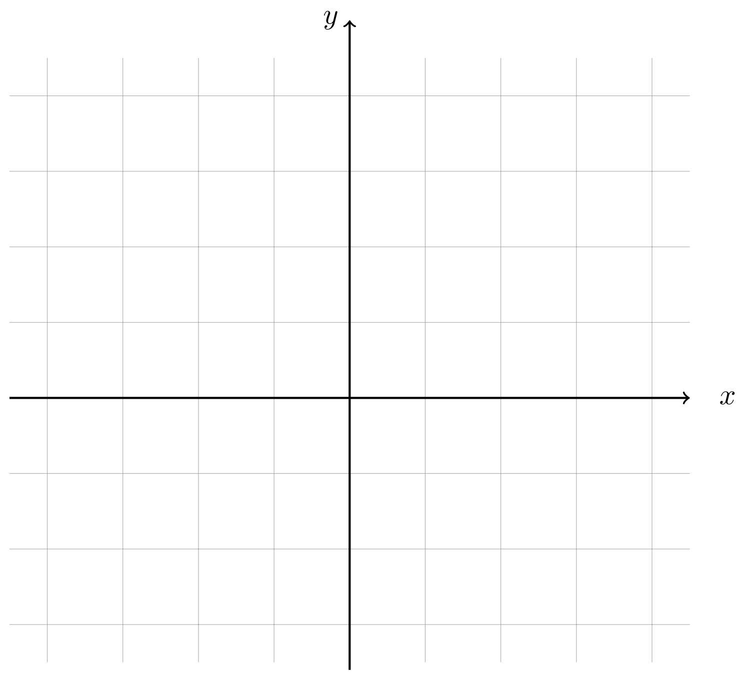
where $\mathcal{R} = [0,1] \times [0,1]$.

Hint: an antiderivative of $\ln x$ *is* $x \ln x - x$.

(3) (*7 points*) Find the volume of the solid bounded by the planes $x = 0$, $y = 2$, $z = 0$, $y = 2x$ and the surface $z = y^2$.

(4) (*3+4=7 points*) Consider the iterated integral $\mathcal{I} = \int_0^3 \int_0^{\sqrt{9-x^2}} (x^2 + y^2)\, dy\, dx$.

(a) Sketch the domain of integration in the xy-plane.



(b) Evaluate $\mathcal{I}$.

(5) (*6 points*) Let $\mathcal{W}$ be a solid with density $\delta(x, y, z)$. Match the following quantities with their expression:

A. Total mass of $\mathcal{W}$

B. y-coordinate of the center of mass

C. Mean value of the density

☐ $\dfrac{\iiint_{\mathcal{W}} x\, dV}{\iiint_{\mathcal{W}} dV}$

☐ $\dfrac{\iiint_{\mathcal{W}} y\, dV}{\iiint_{\mathcal{W}} dV}$

☐ $\dfrac{\iiint_{\mathcal{W}} z\, dV}{\iiint_{\mathcal{W}} dV}$

☐ $\displaystyle\iiint_{\mathcal{W}} 1\, dV$

☐ $\displaystyle\iiint_{\mathcal{W}} \delta(x, y, z)\, dV$

☐ $\dfrac{\iiint_{\mathcal{W}} \delta(x, y, z)\, dV}{\iiint_{\mathcal{W}} dV}$

☐ $\dfrac{\iiint_{\mathcal{W}} x\, \delta(x, y, z)\, dV}{\iiint_{\mathcal{W}} \delta(x, y, z)\, dV}$

☐ $\dfrac{\iiint_{\mathcal{W}} y\, \delta(x, y, z)\, dV}{\iiint_{\mathcal{W}} \delta(x, y, z)\, dV}$

☐ $\dfrac{\iiint_{\mathcal{W}} z\, \delta(x, y, z)\, dV}{\iiint_{\mathcal{W}} \delta(x, y, z)\, dV}$

(6) (*6 points*) Let $\mathcal{D}$ be the domain bounded by the curves $y = x$, $y = 3x$, $xy = 1$ and $xy = 3$ in the first quadrant. What does the integral $\iint_{\mathcal{D}} xy\,dA$ become under the change of variables $x = \dfrac{u}{v}$, $y = v$?

☐ $\displaystyle\int_1^3 \int_1^3 u\,dv\,du.$

☐ $\displaystyle\int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} u\,dv\,du.$

☐ $\displaystyle\int_1^3 \int_{u^2}^{3u^2} u\,dv\,du.$

☐ $\displaystyle\int_1^3 \int_1^3 \frac{u}{v}\,dv\,du.$

☐ $\displaystyle\int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} \frac{u}{v}\,dv\,du.$

☐ $\displaystyle\int_1^3 \int_{u^2}^{3u^2} \frac{u}{v}\,dv\,du.$

(7) *(3+4=7 points)* Let Γ be the curve parametrized by $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + e^t\mathbf{k}$ for $0 \leq t \leq 1$.

(a) Determine the starting point and the end point of Γ.

(b) Consider the function $f(x, y, z) = x\sqrt{y}\ln(1 + x - y + z)$ and let $\mathbf{F} = \nabla f$.
Evaluate $\displaystyle\int_\Gamma \mathbf{F} \cdot d\mathbf{r}$.

(8) (*6 points*) Let Σ be the part of the sphere with equation $x^2+y^2+z^2=1$ in the $x<0$ region. The integral $\displaystyle\iint_\Sigma x\,dS$ is equal to...

☐ $\displaystyle\int_0^\pi\int_0^\pi \cos\theta\sin\varphi\,d\theta\,d\varphi.$

☐ $\displaystyle\int_0^\pi\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos\theta\sin\varphi\,d\theta\,d\varphi.$

☐ $\displaystyle\int_0^\pi\int_0^{2\pi} \cos\theta\sin\varphi\,d\theta\,d\varphi.$

☐ $\displaystyle\int_0^\pi\int_0^{2\pi} \cos\theta\sin^2\varphi\,d\theta\,d\varphi.$

☐ $\displaystyle\int_0^\pi\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos\theta\sin^2\varphi\,d\theta\,d\varphi.$

☐ $\displaystyle\int_0^\pi\int_0^\pi \cos\theta\sin^2\varphi\,d\theta\,d\varphi.$

☐ $\displaystyle\int_0^\pi\int_0^{2\pi}\int_0^1 \cos\theta\sin^2\varphi\,\rho\,d\rho\,d\theta\,d\varphi.$

☐ $\displaystyle\int_0^\pi\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_0^1 \cos\theta\sin^2\varphi\,\rho\,d\rho\,d\theta\,d\varphi.$

☐ $\displaystyle\int_0^\pi\int_0^\pi\int_0^1 \cos\theta\sin^2\varphi\,\rho\,d\rho\,d\theta\,d\varphi.$

(9) (*6 points*) Let Σ be the surface parametrized by

$$S(u,v) = \big(u^2\,,\, v^2\,,\, u+2v\big)$$

for $0 \leq u \leq 2$ and $0 \leq v \leq 2$.

Find an equation for the plane tangent to Σ at the point $(1,1,3)$.

Present your answer in the form $ax + by + cz + d = 0$ *(with* a*,* b*,* c*,* d *to be determined).*

(10) (*7 points*) Evaluate $\int_{\Sigma} \mathbf{F} \cdot d\mathbf{S}$ where

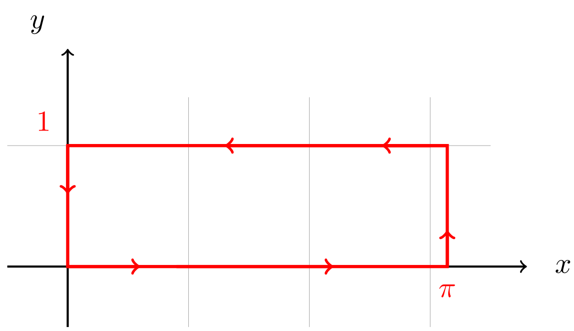
$$\mathbf{F}(x, y, z) = \langle xy\,,\, 4x^2, yz \rangle$$

and Σ is the surface $z = xe^y$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$, oriented in the positive x-direction.

(11) (*7 points*) Calculate the integral of the vector field

$$\mathbf{F}(x,y) = \left\langle y^3 + 2xy + 2y - e^{-x^2}\,,\ x^2 + 3xy^2 + 5x + \cos(\sqrt{y}) \right\rangle$$

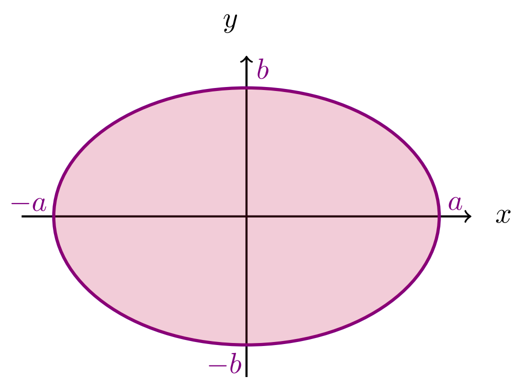
along the closed rectangular path drawn below.



(12) (*7 points*) Use Green's Theorem to calculate the area of the full elliptic domain with equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \leq 1$.

Hint: it might be useful to observe that ***one*** *of the following is a parametrization of the ellipse (for* $0 \leq t \leq 2\pi$*)*

$$\begin{cases} x(t) = a\cos t \\ y(t) = b\sin t \end{cases} \qquad\qquad \begin{cases} x(t) = \dfrac{\cos t}{a} \\ y(t) = \dfrac{\sin t}{b} \end{cases}.$$



(13) (*7 points*) Let $\mathbb{S}$ be the hemisphere $x^2 + y^2 + z^2 = 9$ in the $z \geq 0$ region, oriented upward and $\mathbf{F}(x, y, z) = 2x \cos z\mathbf{i} + e^x \sin z\mathbf{j} + xe^y\mathbf{k}$. Evaluate $\displaystyle\iint_{\mathbb{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

(14) (*5 points*) Let Σ denote the part of the paraboloid $y = x^2 + z^2$ with $0 \leq y \leq 1$. Consider

$$\mathbf{F}(x, y, z) = \left\langle z\sqrt{7} + y + e^{x^2}\ ,\ xy + \sin\sqrt{y}\ ,\ 0 \right\rangle.$$

Evaluate $\displaystyle\iint_\Sigma \operatorname{curl}\mathbf{F} \cdot d\mathbf{S}$, using normal vectors pointing in the positive y-direction.

Hint: it might help to use surface invariance.

(15) (*5 points*) Let $\mathbf{F}$ be a vector field defined on a simple solid region $\mathcal{W}$ with boundary the closed surface $\mathcal{S}$. Then, $\iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ is...

☐ a vector.

☐ $\iiint_{\mathcal{W}} \operatorname{div} \mathbf{F} \, dV$.

☐ 0 be the Divergence Theorem.

☐ 0 by Stokes Theorem.

☐ not well-defined.

☐ none of the above.

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